Perfect Pavelka Logic

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- Zadeh introduced his Fuzzy Sets in 1965.
- In 1968–9 Goguen outlined some characteristic features fuzzy logic should obey; in his article The logic of inexact concepts he game to a conclusion that complete residuated lattices should have a similar role to fuzzy logic than Boolean algebras have to Classical Logic.
- In 1979 Pavelka published a series of articles On Fuzzy Logic I, II, III, in which he discussed the matter in depth. This meant a generalization of Classical Logic in such a way that axioms, theories, theorems, and tautologies need not be only fully true or fully false, but may be also true to a degree and, therefore, giving rise to such concepts as fuzzy theories, fuzzy set of axioms, many-valued rules of inference, provability degree, truth degree, fuzzy consequence operation etc.

Pavelka's definitions and concepts are meaningful in any fixed complete residuated lattice L. Given L-valued (fuzzy sub-)sets X, Y, a fuzzy consequence operation $\mathcal{C}$ satisfies

- $X \leq \mathcal{C}(X)$,
- if $X \leq Y$ then $\mathcal{C}(X) \leq \mathcal{C}(Y)$,
- $\mathcal{C}(X) = \mathcal{C}(\mathcal{C}(X))$.

The main question if: how to define a semantic consequence operation $\mathcal{C}^{sem}$ and a syntactic consequence operation $\mathcal{C}^{syn}$ and when do they coincide, i.e.

$$\mathcal{C}^{sem}(X)(\alpha) = \mathcal{C}^{syn}(X)(\alpha) \text{ for all } X \text{ and all } \alpha \in X.$$

Pavelka in 1979: If $L = [0,1]$ the answer is affirmative iff L is an MV-algebra. Turunen in 1995: the answer is affirmative if L is an injective MV-algebra. Turunen in 2013: the answer is affirmative iff L is a complete MV-algebra.

The set of atomic formulas $\mathcal{F}_0$ is composed of propositional variables $\mathtt{p}, \mathtt{q}, \mathtt{r}, \mathtt{s}, \cdots$ and truth constants $\mathbf{a}$ corresponding to elements $a \in L$; they generalize the classical truth constants $\bot$ and $\top$. The set $\mathcal{F}$ of all formulas is then constructed in the usual way. Any mapping $v : \mathcal{F}_0 \to L$ such that $v(\mathbf{a}) = a$ for all truth constants $\mathbf{a}$ can be extended recursively into the whole $\mathcal{F}$ by setting

$$v(\alpha \,\mathtt{imp}\, \beta) = v(\alpha) \to v(\beta) \qquad \text{and}$$
$$v(\alpha \,\mathtt{and}\, \beta) = v(\alpha) \odot v(\beta).$$

Such mappings v are called valuations. The truth degree of a wff α is the infimum of all values $v(\alpha)$, that is

$$\mathcal{C}^{\mathrm{sem}}(\alpha) = \bigwedge\{v(\alpha) \mid v \text{ is a valuation }\}.$$

We may also fix some set $\mathcal{T} \subseteq \mathcal{F}$ of wffs and associate to each $\alpha \in \mathcal{T}$ a value $\mathcal{T}(\alpha)$ determining its degree of truth. We consider valuations v such that $\mathcal{T}(\alpha) \leq v(\alpha)$ for all wffs α. If such a valuation exists, then $\mathcal{T}$ is called satisfiable and v satisfies $\mathcal{T}$. We say that $\mathcal{T}$ is a fuzzy theory and the corresponding formulae α are the special axioms Then we consider values

$$\mathcal{C}^{\text{sem}}(\mathcal{T})(\alpha) = \bigwedge\{v(\alpha) \mid v \text{ is a valuation, } v \text{ satisfies } \mathcal{T}\}.$$

The set of logical axioms in Pavelka's Fuzzy Logic, denoted by $\mathtt{A}$, is composed by the following eleven forms of formulae; they receive the value $\mathbf{1}$ in any valuation v (except (Ax. 7))

(Ax. 1)	$\alpha \;\mathtt{imp}\; \alpha$,
(Ax. 2)	$(\alpha \;\mathtt{imp}\; \beta) \;\mathtt{imp}\; [(\beta \;\mathtt{imp}\; \gamma) \;\mathtt{imp}\; (\alpha \;\mathtt{imp}\; \gamma)]$,
(Ax. 3)	$(\alpha_1 \;\mathtt{imp}\; \beta_1) \;\mathtt{imp}\; \{(\beta_2 \;\mathtt{imp}\; \alpha_2) \;\mathtt{imp}\; [(\beta_1 \;\mathtt{imp}\; \beta_2) \;\mathtt{imp}\; (\alpha_1 \;\mathtt{imp}\; \alpha_2)]\}$,
(Ax. 4)	$\alpha \;\mathtt{imp}\; \mathbf{1}$,
(Ax. 5)	$\mathbf{0} \;\mathtt{imp}\; \alpha$,
(Ax. 6)	$(\alpha \;\mathsf{and}\; \mathtt{not}\; \alpha) \;\mathtt{imp}\; \beta$,
(Ax. 7)	$\mathbf{a}$,
(Ax. 8)	$\alpha \;\mathtt{imp}\; (\beta \;\mathtt{imp}\; \alpha)$,
(Ax. 9)	$(\mathbf{1} \;\mathtt{imp}\; \alpha) \;\mathtt{imp}\; \alpha$,
(Ax. 10)	$[(\alpha \;\mathtt{imp}\; \beta) \;\mathtt{imp}\; \beta] \;\mathtt{imp}\; [(\beta \;\mathtt{imp}\; \alpha) \;\mathtt{imp}\; \alpha]$,
(Ax. 11)	$(\mathtt{not}\; \alpha \;\mathtt{imp}\; \mathtt{not}\; \beta) \;\mathtt{imp}\; (\beta \;\mathtt{imp}\; \alpha)$.

A fuzzy rule of inference is a scheme

$$\frac{\alpha_1, \cdots, \alpha_n}{r^{\mathrm{syn}}(\alpha_1, \cdots, \alpha_n)} \quad , \quad \frac{a_1, \cdots, a_n}{r^{\mathrm{sem}}(a_1, \cdots, a_n)}$$

where the wffs $\alpha_1, \cdots, \alpha_n$ are premises and the wff $r^{\mathrm{syn}}(\alpha_1, \cdots, \alpha_n)$ is the conclusion. The values $a_1, \cdots, a_n$ and $r^{\mathrm{sem}}(a_1, \cdots, a_n) \in L$ are the corresponding truth values. The mappings $r^{\mathrm{sem}} : L^n \to L$ are semi-continuous, i.e.

$$r^{\mathrm{sem}}(a_1, \cdots, \bigvee_{j \in \Gamma} a_{k_j}, \cdots, a_n) = \bigvee_{j \in \Gamma} r^{\mathrm{sem}}(a_1, \cdots, a_{k_j}, \cdots, a_n) \quad (1)$$

holds for all $1 \leq k \leq n$. Moreover, the fuzzy rules are required to be sound in the sense that

$$r^{\mathrm{sem}}(v(\alpha_1), \cdots, v(\alpha_n)) \leq v(r^{\mathrm{syn}}(\alpha_1, \cdots, \alpha_n))$$

holds for all valuations v.

REMARK 1 *The semi-continuity condition (1) can be replaced without any dramatic consequences by isotonicity condition (which is a weaker condition): if* $a_k \leq b_k$, *then*

$$r^{\mathrm{sem}}(a_1, \cdots, a_k, \cdots, a_n) \leq r^{\mathrm{sem}}(a_1, \cdots, b_k, \cdots, a_n) \tag{2}$$

for each index $1 \leq k \leq n$.

The following Pavelka's fuzzy rules of inference, a set $\mathtt{R}$.

Generalized Modus Ponens:

$$\frac{\alpha, \alpha \;\mathtt{imp}\; \beta}{\beta} \quad , \quad \frac{a, b}{a \odot b}$$

a-Consistency testing rules:

$$\frac{\mathbf{a}}{\mathbf{0}} \quad , \quad \frac{b}{c}$$

where **a** is a truth constant and $c = \mathbf{0}$ if $b \leq a$ and $c = \mathbf{1}$ otherwise.

a-Lifting rules:

$$\frac{\alpha}{\mathbf{a} \;\mathtt{imp}\; \alpha} \quad , \quad \frac{b}{a \rightarrow b}$$

where **a** is a truth constant.

Rule of Bold Conjunction:

$$\frac{\alpha, \beta}{\alpha \;\mathtt{and}\; \beta} \quad , \quad \frac{a, b}{a \odot b}$$

It is easy to see that also a Rule of Bold Disjunction (not included in the list of Pavelka)

$$\frac{\alpha, \beta}{\alpha \texttt{ or } \beta} \quad , \quad \frac{a, b}{a \oplus b}$$

is a rule of inference in Pavelka's sense. Indeed, isotonicity of r^{sem} follows by the isotonicity of the MV-operation $\oplus$ and soundness can be verified by taking a valuation v and observing that

$$\begin{aligned} r^{\mathrm{sem}}(v(\alpha), v(\beta)) &= v(\alpha) \oplus v(\beta) \\ &= v(\alpha \texttt{ or } \beta) \\ &= v(r^{\mathrm{syn}}(\alpha, \beta)). \end{aligned}$$

This rule will be essential in Perfect Pavelka Logic.

A meta proof (called R-proof by Pavelka) w of a wff α in a fuzzy theory $\mathcal{T}$ is a finite sequence

$$\begin{array}{ccl} \alpha_1 & , & a_1 \\ \vdots & & \vdots \\ \alpha_m & , & a_m, \text{ the degree of the meta proof } w \end{array}$$

(i) $\alpha_m = \alpha$,
(ii) for each i, $1 \leq i \leq m$, α_i is a logical axiom, or is a special axiom of a fuzzy theory $\mathcal{T}$, or there is a fuzzy rule of inference and well formed formulae $\alpha_{i_1}, \cdots, \alpha_{i_n}$ with $i_1, \cdots, i_n < i$ such that $\alpha_i = r^{\mathrm{syn}}(\alpha_{i_1}, \cdots, \alpha_{i_n})$,
(iii) for each i, $1 \leq i \leq m$, the value $a_i \in L$ is given by

$$a_i = \begin{cases} a & \text{if } \alpha_i \text{ is the truth constant axiom } \mathbf{a}, \\ \mathbf{1} & \text{if } \alpha_i \text{ is some other logical axiom in the set } \mathtt{A}, \\ \mathcal{T}(\alpha_i) & \text{if } \alpha_i \text{ is a special axiom of a fuzzy theory } \mathcal{T}, \\ r^{\mathrm{sem}}(a_{i_1}, \cdots, a_{i_n}) & \text{if } \alpha_i = r^{\mathrm{syn}}(\alpha_{i_1}, \cdots, \alpha_{i_n}). \end{cases}$$

Since a wff α may have various meta proofs with different degrees, we define the provability degree of a formula α to be the supremum of all such values, i.e.,

$$\mathcal{C}^{\mathrm{syn}}(\mathcal{T})(\alpha) = \bigvee\{a_m \mid w \text{ is a meta proof for } \alpha \text{ in } \mathcal{T}\}.$$

In particular, $\mathcal{C}^{\text{syn}}(\mathcal{T})(\alpha) = \mathbf{0}$ means that either α does not have any meta proof or that for any meta proof w of α the value $a_m = \mathbf{0}$. A fuzzy theory $\mathcal{T}$ is consistent if $\mathcal{C}^{\text{sem}}(\mathcal{T})(\mathbf{a}) = a$ for all truth constants $\mathbf{a}$. Any satisfiable fuzzy theory is consistent. Completeness of Pavelka's Sentential Logic:

If $\mathcal{T}$ is consistent, then $\mathcal{C}^{\text{sem}}(\mathcal{T})(\alpha) = \mathcal{C}^{\text{syn}}(\mathcal{T})(\alpha)$ for any wff α.

Thus, in Pavelka's Fuzzy Sentential Logic we may talk about theorems of a degree a and tautologies of a degree b for $a, b \in L$, and these two values coincide for any formula α.

An MV-algebra called Chang's MV-algebra, introduced in 1958, is obtained by considering the following set $\mathcal{C}$ of formal symbols:

$$\mathbf{0}, c, 2c, 3c, \cdots, nc, \cdots \quad \cdots, 1-nc, \cdots, 1-3c, 1-2c, 1-c, \mathbf{1}$$

and then defining the MV-operations as follows

$$\begin{aligned}
\text{if } x = nc \text{ and } y = mc, &\quad \text{then} \quad x \oplus y := (n+m)c,\\
\text{if } x = 1-nc \text{ and } y = 1-mc, &\quad \text{then} \quad x \oplus y := \mathbf{1},\\
\text{if } x = nc \text{ and } y = 1-mc \text{ and } m \leq n, &\quad \text{then} \quad x \oplus y := \mathbf{1},\\
\text{if } x = nc \text{ and } y = 1-mc \text{ and } n < m, &\quad \text{then} \quad x \oplus y := 1-(m-n)c,\\
\text{if } x = 1-mc \text{ and } y = nc \text{ and } m \leq n, &\quad \text{then} \quad x \oplus y := \mathbf{1},\\
\text{if } x = 1-mc \text{ and } y = nc \text{ and } n < m, &\quad \text{then} \quad x \oplus y := 1-(m-n)c,\\
\text{if } x = nc, &\quad \text{then} \quad x^* := 1-nc,\\
\text{if } x = 1-nc, &\quad \text{then} \quad x^* := nc,
\end{aligned}$$

We construct Chang's MV-algebra in a way which better reflects the logical structure we have in mind, it is also more easily visualized. Recall a Product algebra P is a BL-algebra which satisfies additional conditions

$$\begin{aligned} x^{**} &\leq (y \odot x \to z \odot x) \to (y \odot z), \\ x \wedge x^{*} &= \mathbf{0} \end{aligned}$$

for all $x, y, z \in P$. A simple example is the product t-norm on the real unit interval; $x \odot y = xy$. Fix an element $t \in P, 0 < t < 1$. Then the set $T = \{t^n \mid n \geq 0\}$ is an infinite decreasing chain

$$\cdots < t^n < \cdots < t^3 < t^2 < t < t^0 = \mathbf{1}.$$

In fact T is a cancellative lattice-ordered monoid. Now reverse the order and rename the elements t^n by f^n as follows

$$\mathbf{0} = f^0 < f < f^2 < f^3 < \cdots < f^n < \cdots$$

Then the set $F = \{f^n \mid n \geq 0\}$ is an infinite increasing chain.

Assuming $f^n < t^n$ for any natural $n \geq 0$ we construct the set $F \cup T$

$$\mathbf{0} < f < f^2 < f^3 < \cdots < f^n < \cdots \quad \cdots < t^n < \cdots < t^3 < t^2 < t < \mathbf{1}.$$

Notice that $F \cap T = \emptyset$ and $F \cup T$ is a lattice that is not complete as $\bigvee F$ and $\bigwedge T$ do not exists in $F \cup T$; however, if a supremum of a subset of the set $F \cup T$ exists, then it is the greatest element of this subset (and conversely). Similarly, if an infimum of a subset of the set $F \cup T$ exists, then it is the smallest element of this subset (and conversely). We now define the operations $\oplus, \odot$ and * on $F \cup T$ as follows: for any $m, n \geq 0$ $(f^n)^* = t^n, (t^n)^* = f^n$ and

$$\begin{aligned}
f^m \oplus f^n &= f^{m+n}, \\
t^m \oplus t^n &= \mathbf{1}, \\
f^m \oplus t^n &= t^{n-m} \text{ if } n > m \text{ and } = \mathbf{1} \text{ otherwise.} \\
t^m \odot t^n &= t^{m+n}, \\
f^m \odot f^n &= \mathbf{0}, \\
t^m \odot f^n &= f^{n-m} \text{ if } n > m \text{ and } = \mathbf{0} \text{ otherwise.}
\end{aligned}$$

Assume the set of truth values is Chang's MV-algebra $\mathcal{C}$ and the truth constants correspond to the elements of $\mathcal{C}$. All the concepts of Pavelka Logic remain meaningful in such a modification with the exception of the definition of completeness; $\mathcal{C}^{\mathrm{sem}}(\mathcal{T})(\alpha)$ or $\mathcal{C}^{\mathrm{syn}}(\mathcal{T})(\alpha)$ may not exists in $\mathcal{C}$. Call such a modification Perfect Pavelka Logic, PPL in short. We set the following definition

A fuzzy theory $\mathcal{T}$ is weakly complete if whenever $\mathcal{C}^{\mathrm{syn}}(\mathcal{T})(\alpha)$ exists then also $\mathcal{C}^{\mathrm{sem}}(\mathcal{T})(\alpha)$ exists and these two values coincide.

Due to the linearity and discrete structure of $\mathcal{C}$ we observe that if the values $\mathcal{C}^{\mathrm{sem}}(\mathcal{T})(\alpha)$ and $\mathcal{C}^{\mathrm{syn}}(\mathcal{T})(\alpha)$ exist, then

$$
\begin{aligned}
\mathcal{C}^{\mathrm{syn}}(\mathcal{T})(\alpha) &= \max\{a_m \mid a_m \text{ is the value of } w \text{ for } \alpha \text{ in } \mathcal{T}\} \in \mathcal{C}, \\
\mathcal{C}^{\mathrm{sem}}(\mathcal{T})(\alpha) &= \min\{v(\alpha) \mid v \text{ is a valuation, } v \text{ satisfies } \mathcal{T}\} \in \mathcal{C}.
\end{aligned}
$$

Thus, there is a meta proof w for α in the fuzzy theory $\mathcal{T}$ with provability degree $a = a_m = \mathcal{C}^{\mathrm{syn}}(\mathcal{T})(\alpha)$ and a valuation v that satisfies $\mathcal{T}$ and $b = v(\alpha) = \mathcal{C}^{\mathrm{sem}}(\mathcal{T})(\alpha)$. In such cases we write $\mathcal{T} \vdash_a \alpha$ and $\mathcal{T} \models_b \alpha$.

Notice that the symbol $\vdash_a$ denotes that the provability degree is exactly a, not at least a. Thus the existence of a proof of α with value a is a weaker condition than $\mathcal{T} \vdash_a \alpha$. Similarly for $\mathcal{T} \models_b \alpha$.

Our aim is to prove that if a fuzzy theory $\mathcal{T}$ in Perfect Pavelka Logic is consistent, then is it weakly complete.

We remark that in PPL, it suffices to introduce just one truth constant, **f** or **t**. All other constants from Chang's MV-algebra can be derived. Indeed, assume the language contains only one truth constant, say **t**. Then for a fixed propositional variable p, define

$\mathbf{0} := (\mathrm{p}\,\mathtt{and}\,\mathtt{not}\,\mathrm{p})$ and $\mathbf{1} := (\mathrm{p}\,\mathtt{or}\,\mathtt{not}\,\mathrm{p})$ similar to Classical Logic,

and for the intermediate truth constants set

$$\mathbf{t^1} := \mathbf{t},\ \mathbf{t^n} := (\mathbf{t^{n-1}}\,\mathtt{and}\,\mathbf{t}) \text{ for } n \geq 2, \text{ and } \mathbf{f^n} := \mathtt{not}\,\mathbf{t^n} \text{ for } n \geq 1.$$

By setting for all valuations v, $v(\mathbf{t}) = t$, it its obvious that $v(\mathbf{t^n}) = t^n$ and $v(\mathbf{f^n}) = f^n$ for all $n \geq 1$. This situation differs from that of Rational Pavelka Logic where infinitely many truth constants are needed.

Axioms of Perfect Pavelka Logic are the schemas (Ax. 1)–(Ax. 11) and the following

(Ax. 12) $[(\alpha\ \texttt{or}\ \alpha)\ \texttt{and}\ (\alpha\ \texttt{or}\ \alpha)]\ \texttt{equiv}\ [(\alpha\ \texttt{and}\ \alpha)\ \texttt{or}\ (\alpha\ \texttt{and}\ \alpha)]$,

(Ax. 13) $[\alpha\ \texttt{or}\ (\texttt{not}\,\alpha\ \texttt{and}\ \beta)]\ \texttt{imp}\ [(\alpha\ \texttt{imp}\ \beta)\ \texttt{imp}\ \beta]$,

(Ax. 14) $\mathbf{a}\ \texttt{imp}\ \mathbf{b}$, .

where α, β are wffs and **a**, **b** are truth constants. In Chang's MV-algebra $\mathcal{C}$ the axioms (Ax. 12) obtain value **1** in all valuations. In any MV-algebra the axioms (Ax. 13) obtain value **1** in all valuations, and axioms (Ax. 14), called book-keeping axioms, obtain a value $a \to b$.

Rules of inference are those of the original Pavelka Logic and the Rule of Bold Disjunction.

On the basis of the choice of the axioms and by soundness condition of rules of inference, a satisfiable fuzzy theory $\mathcal{T}$ is sound; if $\mathcal{T} \vdash_a \alpha$ and $\mathcal{T} \models_b \alpha$ exist, then $a \leq b$. Our aim is to construct a valuation v such that $v(\alpha) = a$.

Our aim is to construct a valuation v such that $v(\alpha) = a$.
First we prove

PROPOSITION 2 *If $\mathcal{T} \vdash_a \alpha$ then $\mathcal{T} \vdash_1 (\mathbf{a}\ \mathtt{imp}\ \alpha)$.*

Since $\vdash (\alpha\ \mathtt{and}\ \beta)\ \mathtt{imp}\ \alpha$ is provable in Łukasiewicz Sentential Logic, it follows that $\mathcal{T} \vdash_1 [(\alpha\ \mathtt{and}\ \beta)\ \mathtt{imp}\ \alpha]$ holds for any fuzzy theory. This fact will be used in the following important new result.

PROPOSITION 3 *If $\mathcal{T}$ is a consistent fuzzy theory and $\mathcal{T} \vdash_a \alpha$, then $\mathcal{T} \vdash_0 (\mathtt{not}\ \mathbf{a}\ \mathtt{and}\ \alpha)$.*

Proof.
If $a = \mathbf{1}$ then the claim is $\mathcal{T} \vdash_0 (\texttt{not}\, \mathbf{1}\, \texttt{and}\, \alpha)$, which holds as otherwise $\mathcal{T}$ can be shown to be inconsistent; indeed, assume $\mathcal{T} \vdash_b (\texttt{not}\, \mathbf{1}\, \texttt{and}\, \alpha)$, where $b > \mathbf{0}$. Then there is the following meta proof for $\mathbf{0}$ in $\mathcal{T}$:

$(\texttt{not}\, \mathbf{1}\, \texttt{and}\, \alpha)\, \texttt{imp}\, \texttt{not}\, \mathbf{1}$	,	$\mathbf{1}$	,	Lukasiewicz logic
$\texttt{not}\, \mathbf{1}\, \texttt{and}\, \alpha$	,	b	,	assumption
$\texttt{not}\, \mathbf{1}$	,	b	,	Generalized Modus Ponens
$\mathbf{1}\, \texttt{imp}\, \mathbf{0}$	,	b	,	abbreviation of $\texttt{not}$
$\mathbf{1}$	,	$\mathbf{1}$	,	axiom (Ax. 7)
$\mathbf{0}$	,	b	,	Generalized Modus Ponens
$\mathbf{0}$	,	$\mathbf{1}$	,	$\mathbf{0}$-Consistency Testing Rule

Let $a \neq \mathbf{1}$ and assume $\mathcal{T} \vdash_b (\texttt{not}\, \mathbf{a}\, \texttt{and}\, \alpha)$ where $b \neq \mathbf{0}$. Then there is the following meta proof for α in $\mathcal{T}$:

$\mathbf{a}$	,	a	,	Ax. 7
$\texttt{not}\, \mathbf{a}\, \texttt{and}\, \alpha$	,	b	,	assumpt.
$\mathbf{a}\, \texttt{or}\, (\texttt{not}\, \mathbf{a}\, \texttt{and}\, \alpha)$	,	$a \oplus b$	,	Rule of BD
$[\mathbf{a}\, \texttt{or}\, (\texttt{not}\, \mathbf{a}\, \texttt{and}\, \alpha)]\, \texttt{imp}\, [(\mathbf{a}\, \texttt{imp}\, \alpha)\, \texttt{imp}\, \alpha]$	,	$\mathbf{1}$	,	Ax. 13
$(\mathbf{a}\, \texttt{imp}\, \alpha)\, \texttt{imp}\, \alpha$	,	$a \oplus b$	,	GMP
$\mathbf{a}\, \texttt{imp}\, \alpha$	,	$\mathbf{1}$	,	assumpt.
α	,	$a \oplus b$	,	GMP

The inequality $a \oplus b > a$ contradicts the assumption $\mathcal{T} \vdash_a \alpha$. Therefore $b = \mathbf{0}$ and the proof is complete.

Second, letting $\mathcal{T}$ be a fixed fuzzy theory and by defining

$$\alpha \equiv \beta \qquad \text{iff} \qquad \mathcal{T} \vdash_1 (\alpha \,\mathtt{imp}\, \beta) \qquad \text{and} \qquad \mathcal{T} \vdash_1 (\beta \,\mathtt{imp}\, \alpha)$$

we obtain a congruence relation; denote the equivalence classes by $|\alpha|$ and by $\mathcal{F}/\equiv$ the set of all equivalence classes. Then we have

PROPOSITION 4 *Define* $|\alpha| \to |\beta| = |\alpha \,\mathtt{imp}\, \beta|$ *and* $|\alpha|^* = |\,\mathtt{not}\,\alpha|$. *Then* $\langle \mathcal{F}/\equiv, \to, {}^*, |\mathbf{1}| \rangle$ *is a Wajsberg algebra and, hence, an MV-algebra.*

The Lindenbaum algebra $\mathcal{F}/\equiv$ is, in fact, a perfect MV-algebra; use Axiom (Ax. 12). Even more can be proved:

PROPOSITION 5 *Assume* $\mathcal{T}$ *is a consistent fuzzy theory. If* $\mathcal{T} \vdash_a \alpha$ *then* $|\alpha| = |\mathbf{a}|$ *in* $\mathcal{F}/\equiv$.

Indeed, by Proposition 3, $|(\mathtt{not}\,\mathbf{a}\,\mathtt{and}\,\alpha)| = |\mathbf{0}|$, which implies $|\alpha| \leq |\,\mathtt{not}(\mathtt{not}\,\mathbf{a})| = |\mathbf{a}|$. The converse follows by Proposition 2.

Proposition 5 has a consequence that $\mathcal{F}/\equiv$ is completely determinated by the truth constants, which in turn are in one-to-one correspondence with the elements of Chang's MV-algebra $\mathcal{C}$. Therefore there is an MV-isomorphism $\kappa : (\mathcal{F}/\equiv) \to \mathcal{C}$ given by $\kappa(|\mathbf{a}|) = a$, in particular $\kappa(|\mathbf{1}|) = \mathbf{1}$. This isomorphism can be proved by using the book-keeping axioms (Ax. 14); for all truth constants $\mathbf{a}, \mathbf{b}$ a formula $(\mathbf{a}\ \mathtt{imp}\ \mathbf{b})$ is an axiom of degree $a \to b$. We observe that in consistent fuzzy theories this implies $\mathcal{T} \vdash_{a \to b} \mathbf{a}\ \mathtt{imp}\ \mathbf{b}$.
Indeed, assume $\mathcal{T} \vdash_c \mathbf{a}\ \mathtt{imp}\ \mathbf{b}$, where $c > a \to b$. Since $c \leq a \to b$ iff $c \odot a \leq b$, the assumption $c > a \to b$ implies $c \odot a \not\leq b$. Then $\mathcal{T}$ can be shown to be inconsistent by the following meta proof:

$\mathbf{a}\ \mathtt{imp}\ \mathbf{b}$	,	c	,	assumpt.
$\mathbf{a}$	,	a	,	Ax. 7
$\mathbf{b}$	,	$a \odot c$	,	GMP
$\mathbf{0}$	,	$\mathbf{1}$	,	by $\mathbf{b}$-CTR

Therefore $\mathcal{T} \vdash_{a \to b} \mathbf{a}\ \mathtt{imp}\ \mathbf{b}$ for a consistent fuzzy theory $\mathcal{T}$.

In particular (recall the abbreviation of $\texttt{not}\,\alpha$) we have $\mathcal{T} \vdash_{a^*} \texttt{not}\,\mathbf{a}$. In Lindenbaum algebra holds $|\mathbf{a}\,\texttt{imp}\,\mathbf{b}| = |\mathbf{a}| \to |\mathbf{b}|$ and, by Proposition 5, $|\mathbf{a}\,\texttt{imp}\,\mathbf{b}| = |\mathbf{d}|$, where $d = a \to b$. Thus $\kappa(|\mathbf{a}\,\texttt{imp}\,\mathbf{b}|) = a \to b = \kappa(|\mathbf{a}|) \to \kappa(|\mathbf{b}|)$. In particular, $\kappa(|\,\texttt{not}\,\mathbf{a}|) = \kappa(|\mathbf{a}^*|) = a^*$.
Let π be the canonical mapping $\pi : \mathcal{F} \to \mathcal{F}/\equiv$. Then $\kappa \circ \pi$ is the valuation in demand; if $\mathcal{T} \vdash_a \alpha$ then $\kappa \circ \pi(\alpha) = \kappa(|\mathbf{a}|) = a$. In conclusion, we can write

PROPOSITION 6 *If a formula α is provable at a degree $a \in \mathcal{C}$ in a consistent fuzzy theory $\mathcal{T}$, then α is also a tautology at a degree a i.e. its truth degree is a.*

In particular, if $\mathcal{T} \vdash_a \alpha$ where a is in F; the 'false part', we have

PROPOSITION 7 *Let $\mathcal{T}$ be a consistent fuzzy theory. A formula α is provable at a degree $a \in F$ if, and only if α is also a tautology at a degree a.*

By the following example we show how Perfect Pavelka Logic extends Classical Propositional Logic and differs from Pavelka's original approach. The task is to study the validity of the following reasoning

If there is no government subsidies of agriculture, then there are government controls of agriculture.

If there are government controls of agriculture, then there is no agricultural depression.

There is either an agricultural depression or overproduction.

As a matter of fact, there is no overproduction.

Therefore, there are government subsidies of agriculture.

Assume the special axioms are true, but only to a degree, say $\mathcal{T}(\texttt{not}\, p\, \texttt{imp}\, q) = t^3$, $\mathcal{T}(q\, \texttt{imp}\, \texttt{not}\, r) = t^2$, $\mathcal{T}(r\, \texttt{or}\, s) = t^4$ and $\mathcal{T}(\texttt{not}\, s) = t$. Thus we have a fuzzy theory in Perfect Pavelka Logic.

There is the following meta proof for p

$(\texttt{not}\ p\ \texttt{imp}\ q)\ \texttt{imp}\ [(q\ \texttt{imp}\ \texttt{not}\ r)\ \texttt{imp}\ (\texttt{not}\ p\ \texttt{imp}\ \texttt{not}\ r)]$	,	1	,	CPL
$\texttt{not}\ p\ \texttt{imp}\ q$	,	t^3	,	SpA
$(q\ \texttt{imp}\ \texttt{not}\ r)\ \texttt{imp}\ (\texttt{not}\ p\ \texttt{imp}\ \texttt{not}\ r)$	,	t^3	,	GMP
$(q\ \texttt{imp}\ \texttt{not}\ r)$	,	t^2	,	SpA
$\texttt{not}\ p\ \texttt{imp}\ \texttt{not}\ r$	,	t^5	,	GMP
$(\texttt{not}\ p\ \texttt{imp}\ \texttt{not}\ r)\ \texttt{imp}\ (r\ \texttt{imp}\ p)$	,	1	,	CPL
$r\ \texttt{imp}\ p$	,	t^5	,	GMP
$r\ \texttt{or}\ s$	,	t^4	,	SpA
$\texttt{not}\ s$	,	t	,	SpA
r	,	t^5	,	by GMTP
p	,	t^{10}	,	by GMP

where we used Generalized Modus Tollendo Ponens. We conclude that p is provable at least to a degree t^{10}.

We conclude that p is provable at least to a degree t^{10}.
Since for a valuation v such that $v(p) = t^{10}$, $v(q) = f^7$, $v(r) = t^5$ and $v(s) = f$ holds $v(\texttt{not}\, p \,\texttt{imp}\, q) = t^{10} \oplus f^7 = t^3$, $v(q \,\texttt{imp}\, \texttt{not}\, r) = t^7 \oplus f^5 = t^2$, $v(r \,\texttt{or}\, s) = t^5 \oplus f = t^4$ and $v(\texttt{not}\, s) = t$, we conclude that the fuzzy theory $\mathcal{T}$ is satisfiable and

$$\mathcal{C}^{\mathrm{sem}}\mathcal{T}(p) = \mathcal{C}^{\mathrm{syn}}\mathcal{T}(p) = t^{10}.$$

We realize that from at least partially true premises the conclusion is also at least partially true in Perfect Pavelka Logic.

This is not the case in the original $[0,1]$-valued Pavelka Logic. Indeed, replace the special axioms by $\mathcal{T}(\texttt{not}\, p \,\texttt{imp}\, q) = 0.7$, $\mathcal{T}(q \,\texttt{imp}\, \texttt{not}\, r) = 0.8$, $\mathcal{T}(r \,\texttt{or}\, s) = 0.6$, and $\mathcal{T}(\texttt{not}\, s) = 0.9$. Then a valuation v such that $v(p) = 0$, $v(q) = 0.7$, $v(r) = 0.5$, and $v(s) = 0.1$ satisfies $\mathcal{T}$ and

$$\mathcal{C}^{\mathrm{sem}}\mathcal{T}(p) = \mathcal{C}^{\mathrm{syn}}\mathcal{T}(p) = 0.$$