Fuzzy Relational Equations

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- Bartl E. and Belohlavek R.
  Hardness of Solving Relational Equations.
  Accepted in *IEEE Transactions on Fuzzy Systems*.

- Bartl E. and Prochazka P.
  Do We Need Minimal Solutions of Fuzzy Relational Equations in Advance?
  Submitted to *IEEE Transactions on Fuzzy Systems*.

Fuzzy relational equations: introduction

- Prof. Elie Sanchez (1944–2014), French mathematician

- Sanchez's seminal paper:
  Sanchez E. 1976.
  Resolution of composite fuzzy relation equations.
  Information and Control 30:38–48.

Fuzzy relational equations: introduction

- we consider:
  L ... lattice of truth degrees (Sanchez: Brouwerian lattice)
  $X \in L^n$ ... unknown unary fuzzy relation (fuzzy set)
  $S \in L^{n \times m}$ ... given fuzzy relation
  $T \in L^m$ ... given fuzzy set
  $\circ$ ... sup-t-norm composition operator (other types are also possible)
- fuzzy relational equation is an expression

$$X \circ S = T$$

- a solution to $X \circ S = T$ is any $R \in L^n$ for which $R \circ S = T$, i.e.

$$\bigvee_{l=1}^{n} (R_l \otimes S_{lj}) = T_j,$$

  where $S_{lj} \in L$ denotes the degree to which l is related to j by S, R_l is the degree to which l belongs to R; similarly for T_j

Application: medical diagnosis

- known fuzzy relations:
  S ... association between diagnoses and symptoms (corpus of medical knowledge)
  T ... symptoms of a patient
- we want to find:
  R ... diagnosis of the patient such that $R \circ S = T$

Projects:

- 1968–2004, University of Vienna's Medical School: CADAIG I, II (Computer Assisted Diagnosis System)
- nowadays, Vienna General Hospital: MedFrame, MONI system (Monitoring of Nosocomial Infections)

Application: rule based fuzzy control

- we suppose:
  Φ ... control function
  $\mathcal{D} = \{\langle S_i, T_i\rangle \mid i \in I\}$ ... incomplete description of Φ using input-output data pairs
- $\mathcal{D}$ can be seen as a list of linguistic control rules:

$$\text{if } \sigma \text{ is } S_i \text{ then } \tau \text{ is } T_i, \quad i \in I,$$

  where σ is input variable, and τ is output variable
- aim: to interpolate Φ, i.e. to find Φ^* such that

$$\Phi^*(S_i) = T_i, \quad i \in I$$

Application: rule based fuzzy control

- controler is realized by fuzzy relation R connecting inputs S_i with outputs T_i via compositional rule of inference
- that is, we try to solve a system of equations

$$X \circ S_i = T_i, \quad i \in I$$

- in practice, solution is given by (Mamdani and Assilian approach)

$$R_{\mathsf{MA}} = \bigcup_{i \in I} (S_i \times T_i)$$

Criteria of solvability

- well-known fundamental theorem providing a condition for solvability

Theorem (Sanchez, 1976)

An equation $X \circ S = T$ has a solution iff $(S \triangleleft T^{-1})^{-1}$ is a solution. If $X \circ S = T$ is solvable then $(S \triangleleft T^{-1})^{-1}$ is its greatest solution.

- what is the relationship between

$$\hat{R} = (S \triangleleft T^{-1})^{-1} \text{ and}$$
$$R_{\mathsf{MA}} = \bigcup_{i \in I} (S_i \times T_i)?$$

Theorem (corollary of some results of Klawonn, 2000)

If all S_i are normal fuzzy sets and $R_{MA} \subseteq \hat{R}$, then R_{MA} is solution of $X \circ S = T$.

Minimal solutions

- solvable equation:

  unique maximal solution $\hat{R}$;
  how many minimal solutions?

- there may be *no* minimal solution but usually there are *variety* of them
- for instance:

$$x \otimes 0.5 = 0.5$$

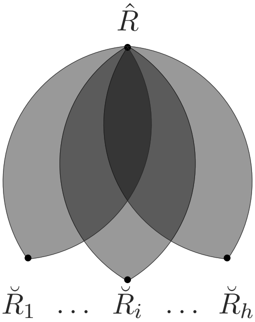
  where $x \in [0,1]$, $\otimes$ is nilpotent minimum defined as

$$a \otimes b = \begin{cases} 0 & \text{if } a + b \leq 1 \\ \min\{a, b\} & \text{otherwise} \end{cases}$$

- this equation has solution-set $(0.5, 1]$, i.e. it has no minimal solution

All solutions

- if there is a minimal solution, the set of all solutions may be represented as the union of intervals bounded from above by the greatest solution and from below by the minimal solutions



- therefore, minimal solutions play a crucial role

Papers on minimal solutions

- due to the importance of minimal solutions, several methods to find all of them have been published
- but more fundamental is the *computational complexity* of finding minimal solutions
- recently, some papers addressing this issue appeared
- all of them adopt the well-known set-cover problem to justify that the problem of finding all minimal solutions is NP-hard

Various flaws in the literature

(i) the notion of *covering* is used in confusing manner

(ii) the *concept of minimal solution* is used in confusing manner

(iii) the *problem of computing all minimal solutions*, presented in the literature as an optimization problem, is ill-conceived since it does not fit the notion of an optimization problem

Recall: Set-cover problem

Set-cover is optimization problem given by:

- instances: pairs $\langle U, \mathcal{S} \rangle$ where $U = \{1, \ldots, m\}$ and $\mathcal{S} = \{C_i \subseteq U \mid i = 1, \ldots, n\}$ such that $\bigcup_{i=1}^{n} C_i = U$
- feasible solution: $\mathcal{C} \subseteq \mathcal{S}$ such that $\bigcup \mathcal{C} = U$
- function sol: assigning to every instance the set of all feasible solutions
- function cost: assigning to every instance $\langle U, \mathcal{S} \rangle$ and every feasible solution $\mathcal{C} \in \mathrm{sol}(U, \mathcal{S})$ a positive rational number specifying the cost of the given solution:

$$\mathrm{cost}(\langle U, \mathcal{S} \rangle, \mathcal{C}) = |\mathcal{C}|$$

- our aim is to minimize the cost

We also require some additional conditions:

- for every instance $\langle U, \mathcal{S} \rangle$, the length of each feasible solution $\mathcal{C} \in \mathrm{sol}(U, \mathcal{S})$ is bounded by a polynomial of the length of $\langle U, \mathcal{S} \rangle$
- cost is computable in polynomial time

Problem to find a minimal solution to fuzzy relational equation

It is sufficient to restrict to a special case: ordinary (Boolean) relational equations.

It is optimization problem given by:

- instances: ordinary equations $X \circ S = T$
- feasible solution: relation R such that $R \circ S = T$
- function sol: assigning to every instance the set of all feasible solutions
- function cost: assigning to every $X \circ S = T$ and every solution $R \in \mathrm{sol}(X \circ S = T)$ the cost of the given solution (next slide)
- our aim is to minimize the cost

Two notions of a minimal solution

A solution $R \in \mathrm{sol}(X \circ S = T)$ is called

- $\#$-*minimal* (cardinality-minimal) if $|R| \leq |R'|$ for every $R' \in \mathrm{sol}(X \circ S = T)$, where $|R| = \sum_{i=1}^{n} R_i$ is the cardinality of R; cost function is then defined by

$$\mathrm{cost}_{\#}(X \circ S = T, R) = |R|$$

- $\subseteq$-*minimal* (inclusion-minimal) if R is minimal w.r.t. $\subseteq$ in $\langle \mathrm{sol}(X \circ S = T), \subseteq \rangle$, i.e. if no R_i may be flipped from 1 to 0 without losing the property of being a solution; cost function is then defined by

$$\mathrm{cost}_{\subseteq}(X \circ S = T, R) = \begin{cases} 1 & \text{if } R \text{ is } \subseteq\text{-minimal} \\ 2 & \text{otherwise} \end{cases}$$

Two Corresponding Optimization Problems

- $\mathrm{MINSOL}_{\#}$ with $\#$-minimal solutions
- $\mathrm{MINSOL}_{\subseteq}$ with $\subseteq$-minimal solutions

Lemma

Function $\mathrm{cost}_{\subseteq}$ *is computable in polynomial time.*

Proof: We have algorithm computing $\mathrm{cost}_{\subseteq}$ in polynomial time ($R[R_i = 0]$ denotes the relation resulting from R by flipping the i-th element to 0):

Input: a solution R to equation $X \circ S = T$
Output: 1 if R is $\subseteq$-minimal; 2 otherwise

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for i = 1, ..., n do
    if R_i = 1 and R[R_i = 0] ∘ S = T then
        return 2
    end if
end for
return 1
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Relationship between set-cover and $\mathrm{MINSOL}_{\ldots}$

Definition

By the *equation associated to* $\langle U, \mathcal{S} \rangle$ (we assume a fixed indexation of elements of U and $\mathcal{S}$) we understand the equation $X \circ S = T$ where $S \in \{0,1\}^{n \times m}$ and $T \in \{0,1\}^m$ are defined by

$$S_{ij} = \begin{cases} 1, & \text{if } j \in C_i, \\ 0, & \text{if } j \notin C_i, \end{cases} \qquad \text{and} \qquad T_j = 1$$

for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

Relationship between set-cover and $\mathrm{MINSOL}_{\ldots}$

Lemma

Let $X \circ S = T$ *be an equation associated to* $\langle U, \mathcal{S} \rangle$ *of set-cover problem. Then*

(a) *the mapping sending an arbitrary* $\mathcal{C} \subseteq \mathcal{S}$ *to* $R_{\mathcal{C}} \in \{0,1\}^n$, *defined by*

$$(R_{\mathcal{C}})_i = 1 \text{ iff } C_i \in \mathcal{C}$$

is a bijection for which

$$\mathcal{C} \in \mathrm{sol}(U, \mathcal{S}) \text{ iff } R_{\mathcal{C}} \in \mathrm{sol}(X \circ S = T)$$

(b) $\mathcal{C} \in \mathrm{opt}_{\#}(U, \mathcal{S})$ *iff* $R_{\mathcal{C}} \in \mathrm{opt}_{\#}(X \circ S = T)$

(c) $\mathcal{C} \in \mathrm{opt}_{\subseteq}(U, \mathcal{S})$ *iff* $R_{\mathcal{C}} \in \mathrm{opt}_{\subseteq}(X \circ S = T)$

- by $\mathrm{opt}_{\ldots}(\ldots)$ we denote the set of all optimal solutions (solutions with minimal cost)

Complexity of MINSOL...

Theorem

(a) $\mathrm{MINSOL}_{\#}$ *is* NP*-hard.*

(b) $\mathrm{MINSOL}_{\subseteq} \in \mathrm{PO}$*.*

Proof:

(a) Directly from NP-hardness of a decision version of set-cover problem.

(b) The following algorithm solves $\mathrm{MINSOL}_{\subseteq}$ and has a polynomial time complexity:

Input: FRE $X \circ S = T$

Output: $\subseteq$-minimal solution to $X \circ S = T$

$R_i \leftarrow 1$ for every $i \in \{1, \ldots, n\}$

while there is $i \in \{1, \ldots, n\}$ such that $(R_i = 1)$ **and** $(R[R_i = 0] \circ S = T)$ **do**

$R \leftarrow R[R_i = 0]$

end while

return R

Problem of computing all $\subseteq$-minimal solutions

- existing papers: $\mathrm{allMINSOL}_{\subseteq}$ is NP-hard optimization problem
- but NP-hardness imply that:

  if P$\neq$NP then there does not exist an efficient algorithm computing all minimal solutions;
- we show a stronger version of this claim is true: condition "if P$\neq$NP" can be dropped
- $\mathrm{allMINSOL}_{\subseteq}$ *is not an optimization problem* in terms of computational complexity theory since there are equations with exponentially many minimal solutions
- original idea: is there any equation such that all $\subseteq$-minimal solutions forms the longest antichain in $\langle\{0,1\}^n, \subseteq\rangle$? (Sperner's theorem)

Problem of computing all $\subseteq$-minimal solutions

Lemma

For every positive integer m, there exist relations $S \in \{0,1\}^{2m \times m}$ and $T \in \{0,1\}^m$ such that the set of all $\subseteq$-minimal solutions of $X \circ S = T$ has 2^m elements.

Proof: Define equation:

$$X \circ \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & \dots & 1 \end{pmatrix}$$

If $R \in \{0,1\}^{2m}$ is a solution, then $R_j = 1$ or $R_{2j} = 1$ or both. If both $R_j = 1$ and $R_{2j} = 1$, then R is not $\subseteq$-minimal. Hence, in a minimal solution R, exactly one of R_j and R_{2j} equals 1. The number of such Rs is clearly 2^m.

Problem of computing all $\subseteq$-minimal solutions

Theorem

There does not exist a polynomial time algorithm solving $\mathrm{allMINSOL}_{\subseteq}$.