Když prvek není roven sám sobě

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Non-existing elements

If x exists then $x = x$.

- proof by contradiction

Non-existing elements

If x exists then $x = x$.

- proof by contradiction

Universe with non-existing elements

Our starting point:

$$x \text{ exists if and only if } x = x$$

Similar approach:

- Dana Scott: Identity and existence in intuitionistic logic (1979)
- Free logic (negative semantics)

Incomplete information

Ignorance of ...

- existence of elements
- equality of elements
- sets membership

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Conditions

L ... *Boolean algebra of conditions* (complete and atomic)
In finite case

- classes of equivalent formulas

$h\colon L \to K$ ... *realities* (complete surjective homomorphisms)
$h\colon L \to 2$ ... *total realities*

$$h(c) = 1 \ \text{... condition } c \text{ is satisfied in reality } h$$

Conditional universes

X with $\approx\colon X \times X \to L$ (*conditional equality*)

$$x \approx y = y \approx x \qquad \text{(symmetry)}$$
$$(x \approx y) \wedge (y \approx z) \leq x \approx z \qquad \text{(transitivity)}$$

- $x \approx y$ ... the condition under which x is equal to y
- $x \approx x$ ... the condition under which x exists

In reality

$h\colon L \to K \ldots$ reality

- $h(x \approx y) = 1 \ldots$ it is satisfied in h that x is equal to y
- $h(x \approx x) = 1 \ldots$ it is satisfied in h that x exists

Separation

X is *separated* if

$$x \approx y = x \approx x = y \approx y \text{ always implies } x = y$$

Ω-valued sets

- Ω-sets, totally fuzzy sets

Ω ... complete Heyting algebra
X with symmetric and transitive $\approx : X \times X \to \Omega$

Equivalence:

- Ω-valued sets
- canonical sheaves on Ω
- sets in Ω-valued universe

Literature

- Denis Higgs: A category approach to boolean-valued set theory (1973)
- Michael Fourman, Dana Scott: Sheaves and logic (1979)
- Robert Goldblatt: Topoi—The Categorial Analysis of Logic (1984)
- Oswald Wyler: Fuzzy logic and categories of fuzzy sets (1995)
- Ulrich Höhle: Fuzzy Sets and Sheaves (2007)

Conditional sets

$A\colon X \to L$
$A(x)$ ... the condition under which x is an element of A
The condition under which A exists:

$$E(A) = \bigwedge_{x \in X} A(x) \to (x \approx x)$$

Subsethood and equality

$$\mathrm{S}_{\approx}(A, B) = \bigwedge_{x \in X} A(x) \to \bigvee_{y \in X} B(y) \wedge (x \approx y)$$

$$A \approx^{+} B = \mathrm{S}_{\approx}(A, B) \wedge \mathrm{S}_{\approx}(B, A)$$

We have

- $E(A) = A \approx^{+} A$ (A exists iff A is equal to A)

Realizations of conditional universes

$\langle X, \approx \rangle \ldots L$-conditional universe
$h \colon L \to K \ldots$ reality

h-realization of $\langle X, \approx \rangle$:

- K-conditional universe $\langle X^h, \approx^h \rangle$
- surjective partial function $X \to X^h$, $x \mapsto x^h$

such that

- X^h is separated
- $x^h \approx^h x^h > 0$ for all $x^h \in X^h$
- if $h(x \approx x) > 0$ then x^h is defined
- $h(x \approx y) = x^h \approx^h y^h$

(it is satisfied in h that x is equal to y iff x is equal to y in h)

Realizations of conditional sets

$A\colon X \to L$
$h\colon L \to K \ldots$ reality

If $h(x \approx x) = 0$ always implies $h(A(x)) = 0$ then we set

$$A^h(x^h) = \bigvee_{\substack{y \in X \\ y^h = x^h}} h(A(y))$$

- $h(A(x)) \leq A^h(x^h) \ldots A$ can lie

(if it is satisfied in h that x is an element of A then x is an element of A in h)

$$h(\mathrm{S}_{\approx}(A, B)) = \mathrm{S}_{\approx^h}(A^h, B^h)$$
$$h(A \approx^+ B) = A^h \approx^{h+} B^h$$

(it is satisfied in h that A is a subset of B iff A is a subset of B in h)

Conditional sets (cont.)

$$\mathrm{C}_{\approx}(A)(x) = \bigvee_{y \in X} A(y) \wedge (x \approx y)$$

Generally: $A \nsubseteq \mathrm{C}_{\approx}(A)$

A is a fixpoint of $\mathrm{C}_{\approx}$ if and only if

- $A \approx^{+} A = 1$
- $A(x) \wedge (x \approx y) \leq A(y)$ (A is compatible with $\approx$)

If A is a fixpoint of $\mathrm{C}_{\approx}$ then

$$h(A(x)) = A^h(x^h)$$

Future work

- conditional sets with $h(A(x)) = A^h(x^h)$
- conditional universe of conditional sets
- realizations with non-existing elements
- concept lattices in this framework