From tennis racket instability to spin squeezing and quantum phase transitions: quantum-classical analogies

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- Symmetric and asymmetric top, tennis racket instability and spin squeezing
- Euler top with a rotor, Lipkin-Meshkov-Glick model and excited state quantum phase transitions
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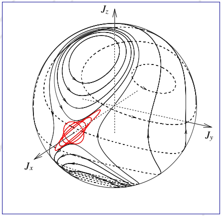
- Summary



Motivation

- "The same equations have the same solutions" [The Feynman Lectures on Physics, Vol. II, Chap. 12-1.]
- Similarity of equations governing spin squeezing and free Euler top. Any deeper relationship?
- Do classical analogues of the Lipkin-Meshkov-Glick exist?

Motion of classical rigid body

Euler dynamic equations

Angular momentum changes by torque

$$\frac{d\vec{L}}{dt} = \vec{M},$$

in a rotating system

$$\frac{d'\vec{A}}{dt} = \frac{d\vec{A}}{dt} - \vec{\omega} \times \vec{A}.$$

Applied for $\vec{L}$

$$\frac{d'\vec{L}}{dt} = \vec{M} - \vec{\omega} \times \vec{L}.$$

L. Euler, “Theoria Motus Corporum Solidorum seu Rigidorum” (1765)

Euler dynamic equations

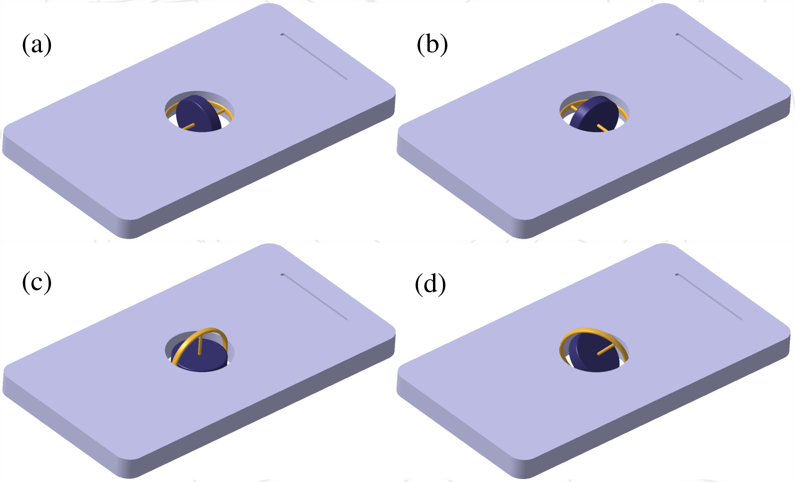
For coordinate system fixed with respect to the rotating body, principal axes of inertia:

$$L_k = I_k \omega_k, \qquad k = 1, 2, 3.$$

For a free top, $\vec{M} = 0$:

$$\begin{aligned}
\dot{L}_1 &= \left(\frac{1}{I_3} - \frac{1}{I_2}\right) L_2 L_3, \\
\dot{L}_2 &= \left(\frac{1}{I_1} - \frac{1}{I_3}\right) L_3 L_1, \\
\dot{L}_3 &= \left(\frac{1}{I_2} - \frac{1}{I_1}\right) L_1 L_2.
\end{aligned}$$

Classical motion of a top with a rotor



Classical motion of a top with a rotor

Euler dynamic equations

Torque stemming from a rotor with axis is fixed with body:

$$\vec{M} = -\vec{M}_{\mathrm{rotor}} = -\frac{d\vec{K}}{dt},$$

$$\begin{aligned} \vec{M} &= -\frac{d'\vec{K}}{dt} - \vec{\omega} \times \vec{K} \\ &= -\vec{\omega} \times \vec{K}, \end{aligned}$$

since $d'\vec{K}/dt = 0$ (the rotor changes neither the magnitude of rotation nor the axis orientation with respect to the rigid body).

This leads to

$$\frac{d'\vec{L}}{dt} = -\vec{\omega} \times \left(\vec{L} + \vec{K}\right).$$

Classical motion of a top with a rotor

Euler dynamic equations

Components of $\vec{L}$:

$$
\begin{aligned}
\dot{L}_1 &= \left(\frac{1}{I_3} - \frac{1}{I_2}\right) L_2 L_3 + \frac{K_2}{I_3} L_3 - \frac{K_3}{I_2} L_2, \\
\dot{L}_2 &= \left(\frac{1}{I_1} - \frac{1}{I_3}\right) L_3 L_1 + \frac{K_3}{I_1} L_1 - \frac{K_1}{I_3} L_3, \\
\dot{L}_3 &= \left(\frac{1}{I_2} - \frac{1}{I_1}\right) L_1 L_2 + \frac{K_1}{I_2} L_2 - \frac{K_2}{I_1} L_1.
\end{aligned}
$$

Classical motion of a top with a rotor

Euler dynamic equations

Suitable to work with the total angular momentum $\vec{J} \equiv \vec{L} + \vec{K}$,

$$\begin{aligned}
\dot{J}_1 &= \left(\frac{1}{I_3} - \frac{1}{I_2}\right) J_2 J_3 + \frac{K_2}{I_2} J_3 - \frac{K_3}{I_3} J_2, \\
\dot{J}_2 &= \left(\frac{1}{I_1} - \frac{1}{I_3}\right) J_3 J_1 + \frac{K_3}{I_3} J_1 - \frac{K_1}{I_1} J_3, \\
\dot{J}_3 &= \left(\frac{1}{I_2} - \frac{1}{I_1}\right) J_1 J_2 + \frac{K_1}{I_1} J_2 - \frac{K_2}{I_2} J_1.
\end{aligned}$$

Euler dynamic equations

These equations conserve kinetic energy and magnitude of the total angular momentum,

$$\dot{E}_{\mathrm{body}} = 0, \qquad \dot{J^2} = 0,$$

where

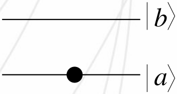
$$\begin{aligned} E_{\mathrm{body}} &= \frac{L_1^2}{2I_1} + \frac{L_2^2}{2I_2} + \frac{L_3^2}{2I_3}, \\ J^2 &= J_1^2 + J_2^2 + J_3^2. \end{aligned}$$

Evolution bound to intersections of energy ellipsoid and a displaced angular momentum sphere ($\vec{J} = \vec{L} + \vec{K}$).

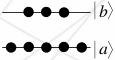
Quantum motion and spin squeezing

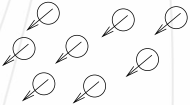
Two bosonic modes $\hat{a}$ and $\hat{b}$ with total number of particles N.
Commutators $[\hat{a}, \hat{a}^\dagger] = [\hat{b}, \hat{b}^\dagger] = 1$.

Introduce operator $\hat{\vec{J}}$:



$$
\begin{aligned}
\hat{J}_x &= \frac{1}{2}(\hat{a}^\dagger \hat{b} + \hat{a}\hat{b}^\dagger), \\
\hat{J}_y &= \frac{1}{2i}(\hat{a}^\dagger \hat{b} - \hat{a}\hat{b}^\dagger), \\
\hat{J}_z &= \frac{1}{2}(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}),
\end{aligned}
$$





with $N = \hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b}$.
Commutation relations:
$[\hat{J}_x, \hat{J}_y] = i\hat{J}_z$, $[\hat{J}_y, \hat{J}_z] = i\hat{J}_x$, and $[\hat{J}_z, \hat{J}_x] = i\hat{J}_y$.

Quantum motion and spin squeezing

Assume a general quadratic Hamiltonian in the form

$$\hat{H} = \sum_{k,l} \chi_{kl} \hat{J}_k \hat{J}_l + \sum_k \Omega_k \hat{J}_k.$$

By a suitable rotation of the coordinate system:

$$\hat{H} = \sum_{k=1}^{3} \left(\chi_k \hat{J}_k^2 + \Omega_k \hat{J}_k \right).$$

Components χ_{kl} form the **twisting tensor**
[T.O., PRA 91, 053826 (2015)].
Coefficients χ_k: eigenvalues of the twisting tensor.

Quantum motion and spin squeezing

The Heisenberg equations of motion for $\hat{H} = \sum_{k=1}^{3} \left(\chi_k \hat{J}_k^2 + \Omega_k \hat{J}_k \right)$:

$$
\begin{aligned}
\frac{d\hat{J}_1}{dt} &= (\chi_2 - \chi_3)(\hat{J}_2\hat{J}_3 + \hat{J}_3\hat{J}_2) + \Omega_2\hat{J}_3 - \Omega_3\hat{J}_2, \\
\frac{d\hat{J}_2}{dt} &= (\chi_3 - \chi_1)(\hat{J}_3\hat{J}_1 + \hat{J}_1\hat{J}_3) + \Omega_3\hat{J}_1 - \Omega_1\hat{J}_3, \\
\frac{d\hat{J}_3}{dt} &= (\chi_1 - \chi_2)(\hat{J}_1\hat{J}_2 + \hat{J}_2\hat{J}_1) + \Omega_1\hat{J}_2 - \Omega_2\hat{J}_1.
\end{aligned}
$$

Correspondence of the models

Equations of the Euler top and of the bosonic modes correspond for

$$\chi_k \leftrightarrow -\frac{1}{2I_k}, \qquad \Omega_k \leftrightarrow \frac{K_k}{I_k},$$

or

$$I_k \leftrightarrow -\frac{1}{2\chi_k}, \qquad K_k \leftrightarrow -\frac{\Omega_k}{2\chi_k}.$$

For angular momenta $\hat{\vec{J}} \leftrightarrow \vec{J}$,
and for energy

$$\hat{H} \leftrightarrow -E_{\mathrm{body}} + \sum_{k=1}^{3} \frac{K_k^2}{2I_k}.$$

Invariance with respect to transformation of χ_k and I_k

Dynamics unchanged if a constant is added to all eigenvalues of χ, i.e.,

$$\chi_k \to \chi_k + \chi_0.$$

Similarly for the moments of inertia:

$$\frac{1}{I_k} \to \frac{1}{I_k} + \frac{1}{I_0}$$

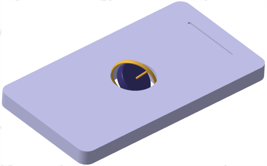
and the angular momentum of the rotor:

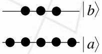
$$K_k \to \frac{K_k}{1 + \frac{I_k}{I_0}}$$

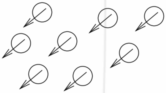
Invariance with respect to transformation of χ_k and I_k

Consequence:

- For each Euler top with a rotor one finds corresponding quadratic collective spin Hamiltonian $\hat{H} = \sum_{k=1}^{3}\left(\chi_k \hat{J}_k^2 + \Omega_k \hat{J}_k\right)$
- For each quadratic collective spin Hamiltonian $\hat{H} = \sum_{k=1}^{3}\left(\chi_k \hat{J}_k^2 + \Omega_k \hat{J}_k\right)$ one finds corresponding Euler top with a rotor.







Invariance with respect to transformation of χ_k and I_k

Proof: mass can be assembled such as to have arbitrary principal moments of inertia I_k, provided they are positive and satisfy the triangle inequality $I_j \leq I_k + I_l$.
First condition: by a suitable choice of the additive constant χ_0 making all values χ_k negative.
If the resulting values I_k violate the triangle inequality (say, $I_1 > I_2 + I_3$), choose I_0 satisfying

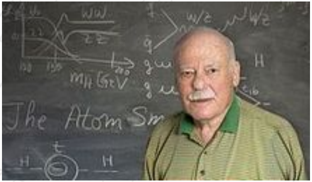
$$0 < I_0 < \frac{I_2 I_3 + \sqrt{I_2^2 I_3^2 + I_1 I_2 I_3 (I_1 - I_2 - I_3)}}{I_1 - I_2 - I_3}.$$

H. J. Lipkin, N. Meshkov, A.J. Glick, "Validity of many-body approximation methods for a solvable model..." Nuclear Physics **62**, 188 (1965).

- Hamiltonian

$$\hat{H} = \epsilon \hat{J}_3 + V(\hat{J}_1^2 - \hat{J}_2^2) + W(\hat{J}_1^2 + \hat{J}_2^2)$$

- N fermions in two degenerate levels whose energies differ by ϵ.
- Exactly solvable (under some conditions).
- Toy model for quantum phase transitions.



Harry (Zvi) Lipkin
(1921 - 2015)

Lipkin-Meshkov-Glick model

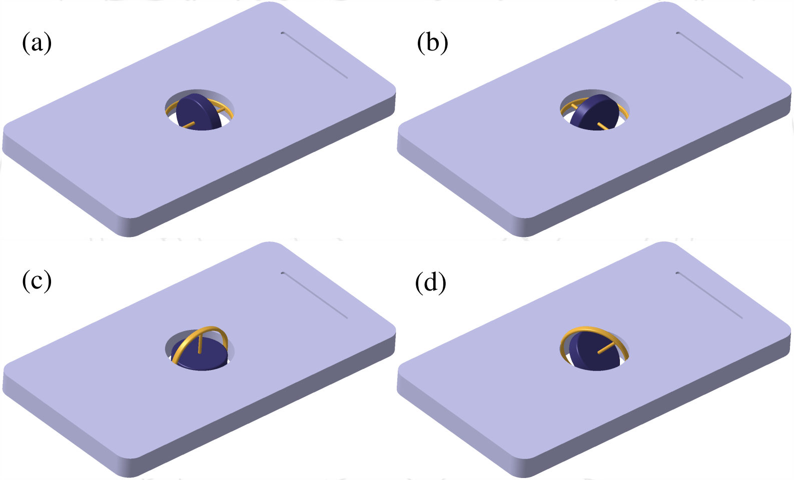
LMG corresponds to a diagonal twisting tensor χ,

$$\chi = \begin{pmatrix} W+V & 0 & 0 \\ 0 & W-V & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and a rotational vector $\vec{\Omega} = (0, 0, \epsilon)$.

- Any diagonal χ can be expressed in a form equivalent to the quadratic term of LMG.
- The LMG parameters are $W = (\chi_1 + \chi_2)/2 - \chi_3$ and $V = (\chi_1 - \chi_2)/2$.
- Any quadratic Hamiltonian is equivalent to LMG, provided the linear term is along a principal axis.
- Any free top with a **rotor along a principal axis** corresponds to a LMG.

Lipkin-Meshkov-Glick model



(a), (b) and (c) correspond to a LMG.
(d) corresponds to a **generalized LMG.**

Free symmetric top and spin squeezing by one-axis twisting

Symmetric top with $I_1 = I_2 \neq I_3$ with no rotor, i.e., $K_k = 0$.

$$\begin{aligned} \dot{\omega}_1 &= -\tilde{\Omega}\omega_2, \\ \dot{\omega}_2 &= \tilde{\Omega}\omega_1, \\ \dot{\omega}_3 &= 0, \end{aligned}$$

where

$$\tilde{\Omega} \equiv \frac{I_3 - I_1}{I_1}\omega_3 = \left(\frac{1}{I_1} - \frac{1}{I_3}\right) J_3.$$

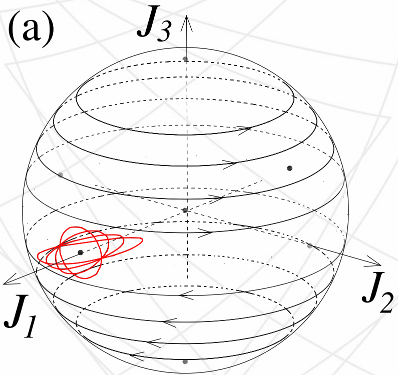
In quantum domain:

$$\hat{H}_{\mathrm{OAT}} = \chi \hat{J}_3^2$$

with

$$\chi = \frac{1}{2I_1} - \frac{1}{2I_3}.$$

Free symmetric top and spin squeezing by one-axis twisting



$$\hat{H}_{\mathrm{OAT}} = \chi \hat{J}_3^2$$

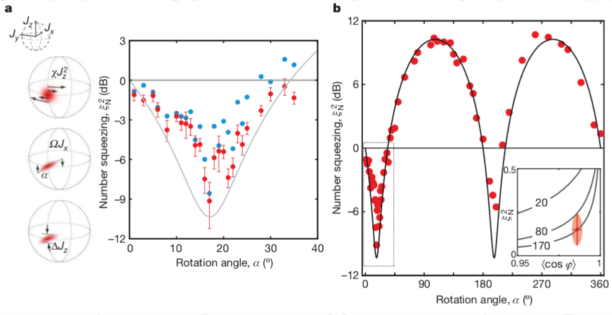
M. Kitagawa and M. Ueda, One-axis-twisting (OAT) scenario of spin squeezing, PRA **47,** 5138 (1993).

Free symmetric top and spin squeezing by one-axis twisting

Pioneering experiments of OAT

Nonlinear atom interferometer surpasses classical precision limit

Gross et al., Nature 464, 1165 (2010)

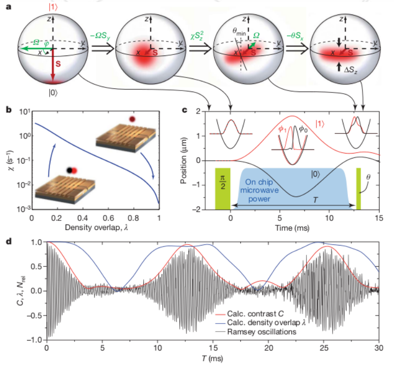


$\sim$ 400 atoms squeezed by ~ -8 dB in $\sim$ 20 ms

Free symmetric top and spin squeezing by one-axis twisting

Pioneering experiments of OAT

Atom-chip-based generation of entanglement for quantum metrology, Riedel et al., Nature 464, 1170 (2010)

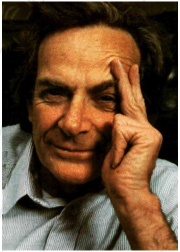


~ 1250 atoms squeezed by ~ -3.7 dB in ~ 10 ms

Free symmetric top and spin squeezing by one-axis twisting

"Surely, You Are Joking, Mr. Feynman!" (1985):

". . . I was in the [Cornell] cafeteria and some guy, fooling around, throws a plate in the air. As the plate went up in the air I saw it wobble, and I noticed the red medallion of Cornell on the plate going around. It was pretty obvious to me that the medallion went around faster than the wobbling. I had nothing to do, so I start to figure out the motion of the rotating plate. I discover that when the angle is very slight, the medallion rotates twice as fast as the wobble rate–two to one. It came out of a complicated equation!"



Free symmetric top and spin squeezing by one-axis twisting

BUT: For a plate $I_1 = I_2 = I_3/2$, therefore

$$\tilde{\Omega} \equiv \frac{I_3 - I_1}{I_1}\omega_3 = \omega_3,$$

with $\tilde{\Omega}$ being the wobble frequency with respect to the (rotating) plate. The wobble frequency with respect to external observer is

$$\tilde{\Omega} + \omega_3 = 2\omega_3.$$

Wobbling is twice as fast as the rotation.
Opposite as in Feynman's story!

[See, e.g. B. F. Chao, Physics Today **42**(2), 15 (1989).]

Free asymmetric top, tennis-racket instability, and two-axis countertwisting

Assume moments of inertia $I_1 < I_3 < I_2$, and $K_k = 0$. Angular velocities evolve as

$$
\begin{aligned}
\dot{\omega}_1 &= \frac{I_2 - I_3}{I_1}\omega_2\omega_3, \\
\dot{\omega}_2 &= \frac{I_3 - I_1}{I_2}\omega_3\omega_1, \\
\dot{\omega}_3 &= \frac{I_1 - I_2}{I_3}\omega_1\omega_2.
\end{aligned}
$$

In quantum domain:

$$\hat{H} = \chi_+ \hat{J}_2^2 - \chi_- \hat{J}_1^2$$

with

$$
\begin{aligned}
\chi_+ &= \frac{1}{2I_3} - \frac{1}{2I_2} \\
\chi_- &= \frac{1}{2I_1} - \frac{1}{2I_3}.
\end{aligned}
$$

Free asymmetric top, tennis-racket instability, and two-axis countertwisting

In the special case of

$$I_3 = \frac{2I_1 I_2}{I_1 + I_2}$$

the Hamiltonian takes the form

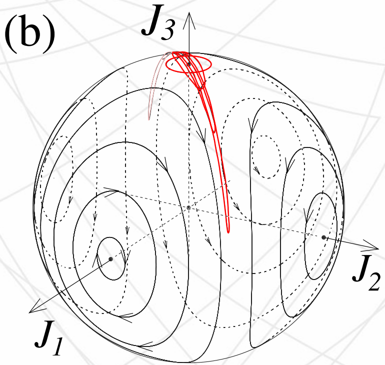
$$\hat{H}_{\mathrm{TACT}} = \chi(\hat{J}_2^2 - \hat{J}_1^2)$$

with

$$\chi = \frac{I_2 - I_1}{4 I_1 I_2}.$$

Two-axis countertwisting (TACT) scenario of spin squeezing.

Free asymmetric top, tennis-racket instability, and two-axis countertwisting



$$\hat{H}_{\mathrm{TACT}} = \chi(\hat{J}_2^2 - \hat{J}_1^2)$$

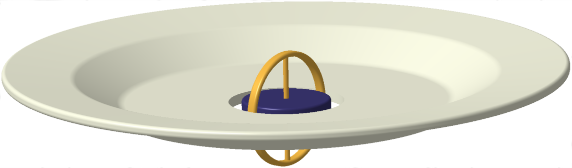
M. Kitagawa and M. Ueda, Two-axis-countertwisting (TACT) scenario of spin squeezing, PRA **47,** 5138 (1993).

Free asymmetric top, tennis-racket instability, and two-axis countertwisting

In classical physics

- Stable rotation around the highest and lowest moment of inertia principal axes.
- Unstable rotation around the intermediate exis.
- Spectacular under zero-gravity conditions: Dzhanibekov effect (Vladimir Dzhanibekov, on 1985 Soyuz T-13 mission).

Symmetric top with a coaxial rotor, spin twisting with coaxial rotation



Quantum motion:

$$\hat{H} = \chi \hat{J}_3^2 + \Omega \hat{J}_3,$$

LMG with $V = 0$.

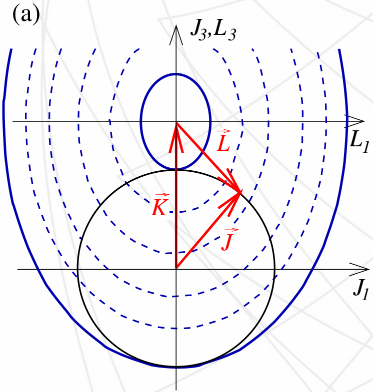
Classical motion, wobble frequency:

$$\tilde{\Omega} = \frac{(I_3 - I_1)\omega_3 + K}{I_1} = \left(\frac{1}{I_1} - \frac{1}{I_3}\right) J_3 + \frac{K}{I_3}.$$

Symmetric top with a coaxial rotor, spin twisting with coaxial rotation

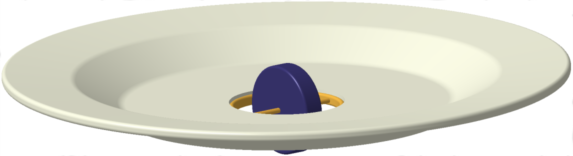
- Wobbling frequency can be tuned by angular momentum of the rotor.
- Example: by choosing $K = -\frac{3}{4} I_3 \omega_3$ one gets the wobbling exactly as in Feynman's story.
- Two different quantum regimes:
  - dominant rotation, $|\chi| N < |\Omega|$, nondegenerate spectrum
  - dominant nonlinearity, $|\chi| N > |\Omega|$, degeneracies occur
- Corresponding classical regimes:
  - dominant rotation, $|K|/J > |1 - I_3/I_1|$, only stable fixed points
  - dominant nonlinearity, $|K|/J < |1 - I_3/I_1|$, unstable fixed points for the rotational axis occur

Symmetric top with a coaxial rotor, spin twisting with coaxial rotation



Angular momentum geometry:
(a) dominant rotation, (b) dominant nonlinearity.

Symmetric top with a perpendicular axis rotor, twist-and-turn Hamiltonian



- Hamiltonian:

$$\hat{H} = \chi \hat{J}_1^2 + \Omega \hat{J}_3,$$

- LMG with $V = W$.
- Twist-and-turn spin squeezing [T.O. PRA 91, 053826 (2015); Muessel et al, PRA 92, 023603 (2015).]

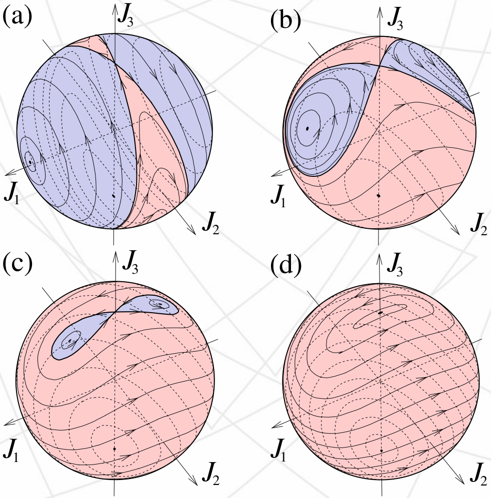
Symmetric top with a perpendicular axis rotor, twist-and-turn Hamiltonian

Two-trap BEC regimes with quantum phase transition at $|\Omega| = N|\chi|$ (Leggett, 2001):

- Rabi regime, $|\Omega/\chi| \gg N$ (population oscillations),
- Josephson regime, $1/N \ll |\Omega/\chi| \ll N$ (oscillations with self-trapping),
- Fock regime, $|\Omega/\chi| \ll 1/N$.

(note $N = 2J$)

Symmetric top with a perpendicular axis rotor, twist-and-turn Hamiltonian



$\Omega/(\chi J) = 0.2$ (a), 1 (b), 1.7 (c), and 2 (d).

Asymmetric top with a principal axis rotor, LMG

Hamiltonian:

$$\hat{H} \quad = \quad \Omega_3 \hat{J}_3 + \sum_{k=1}^{3} \chi_k \hat{J}_k^2,$$

or equivalently

$$\hat{H}_{\mathrm{LMG}} = \epsilon \hat{J}_3 + V(\hat{J}_1^2 - \hat{J}_2^2) + W(\hat{J}_1^2 + \hat{J}_2^2).$$

Asymmetric top with a principal axis rotor, LMG

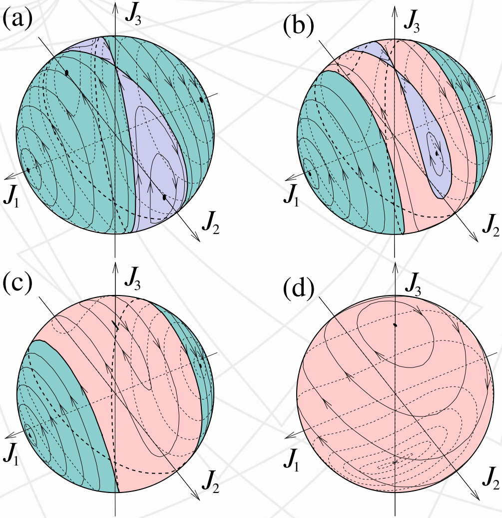
Classical questions:

- How does the rotational axis move with respect to the body?
- What are the fixed points of the rotational axis?
- What is their stability?

Quantum questions:

- What is the spectrum of the Hamiltonian?
- What are the singularities in the spectrum?
- What are the quantum phase transitions due to parameter variations?

Asymmetric top with a principal axis rotor, LMG



$\chi_3 = 0$, $\chi_1 = -10\chi_2$,
(a) $\Omega_3 = 0$, (b) $|\Omega_3| = 1.7J|\chi_2|$, (c) $|\Omega_3| = 2J|\chi_2|$, (d) $|\Omega_3| = 2J|\chi_1|$.

Asymmetric top with a principal axis rotor, LMG

Stationary values of the angular momentum:

$$\mathrm{grad}\ E_{\mathrm{body}} = \lambda\ \mathrm{grad}\ J^2$$

leads to

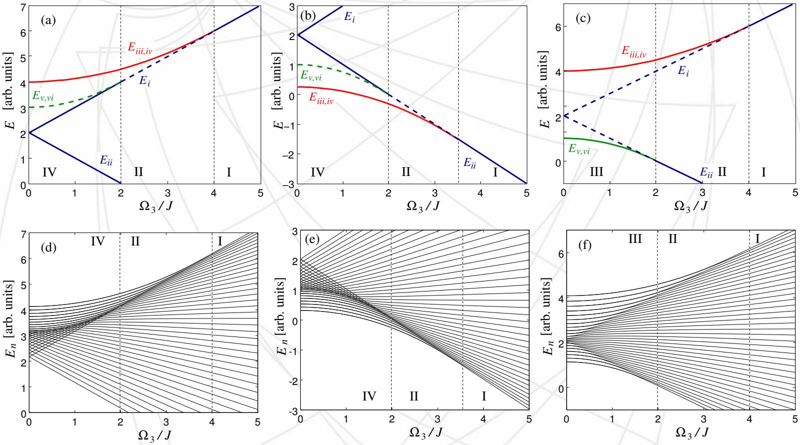
$$\begin{aligned} J_1 &= \frac{I_3 K_1 J_3}{(I_3 - I_1)J_3 + I_1 K_3}, \\ J_2 &= \frac{I_3 K_2 J_3}{(I_3 - I_2)J_3 + I_2 K_3}, \end{aligned}$$

and to the polynomial equation for J_3,

$$\sum_{n=0}^{6} a_n J_3^n = 0,$$

with an explicit expression for coefficients a_n.

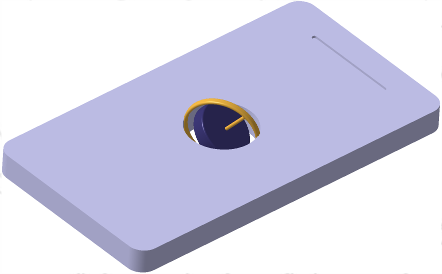
Asymmetric top with a principal axis rotor, LMG



Energies of the stationary angular momenta and spectra of the Hamiltonian with $N = 40$. (a), (d): $\chi_1 = 4$, $\chi_2 = 3$, $\chi_3 = 2$, (b), (e): $\chi_1 = 0.25$, $\chi_2 = 1$, $\chi_3 = 2$, (c), (f) $\chi_1 = 1$, $\chi_2 = 4$, $\chi_3 = 2$.

Excited state quantum phase transitions

Asymmetric top with a general axis rotor, generalized LMG

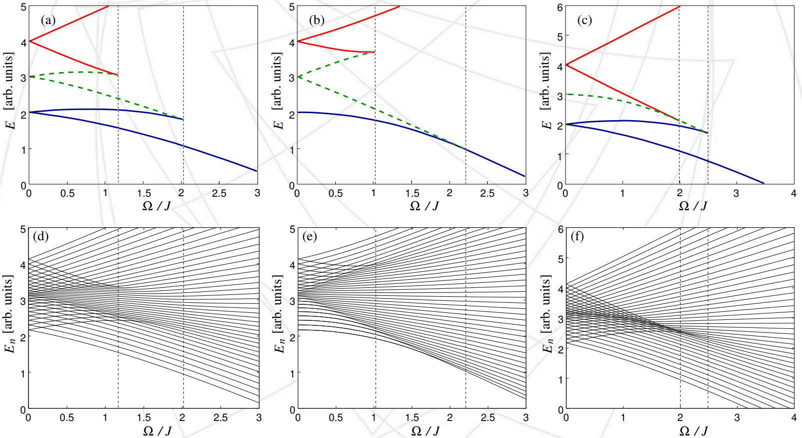


Stationary values of the angular momentum:

$$\mathrm{grad}\ E_{\mathrm{body}} = \lambda\ \mathrm{grad}\ J^2$$

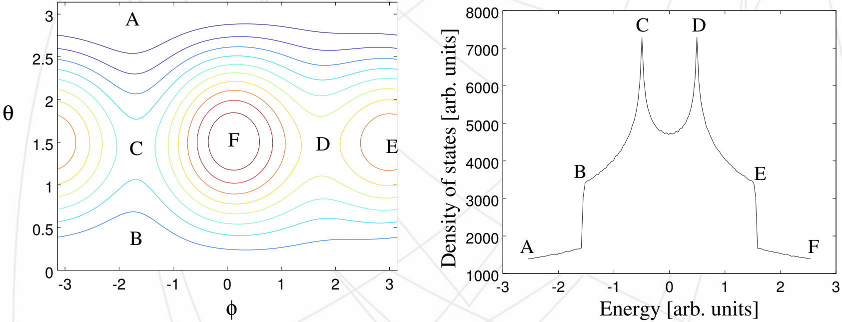
Points of touch of the energy ellipsoid and the angular momentum sphere.

Asymmetric top with a general axis rotor, generalized LMG



The twisting tensor eigenvalues $\chi_1 = 4$, $\chi_2 = 3$, $\chi_3 = 2$, the ratio of components of vector $\vec{\Omega}$ are $\Omega_1 : \Omega_2 : \Omega_3$ as follows, (a,d) 2:1:1, (b,e) 1:2:0, (c,f) 2:0:1.

Asymmetric top with a general axis rotor, generalized LMG

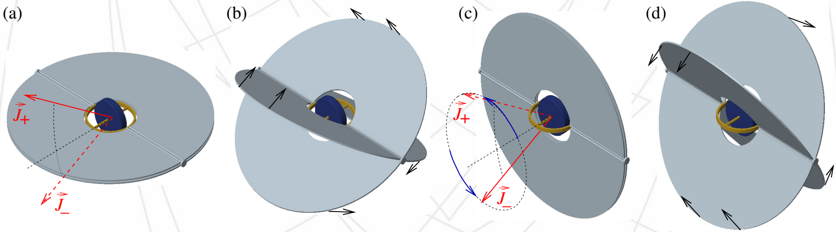


Contours of the equal energy (left) and density of states (right) with $(\chi_1, \chi_2, \chi_3) = (2, 0, -2)$ and $(\Omega_1, \Omega_2, \Omega_3) = (0.5, 0.5, 0.5)$.

Time crystals:

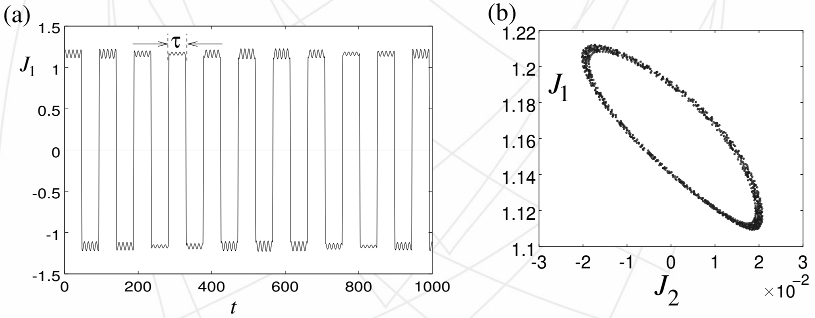
- Concept introduced by F. Wilczek (2012), processes in which spontaneous breaking of time symmetry occurs
- Floquet time crystal: Hamiltonian periodic with τ but dynamics repeats with period $n\tau$; robust, long lasting
- Experiments 2017, Nature: trapped ions (Monroe group), diamond NV centers (Lukin group). Disorder-induced many-body localization.
- Floquet time crystal in "clean system" (no disorder induced MBL): Russomanno et al, PRB **95,** 214307 (2017). Switching parameters of the LMG.

Classical realization:



LMG Floquet time crystals

Classical realization:



Conclusion

Problems of classical physics

- gyroscope motion,
- satellite stabilization,
- Earth wobble, etc.

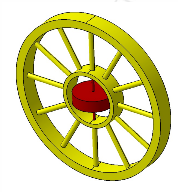
closely related to quantum problems

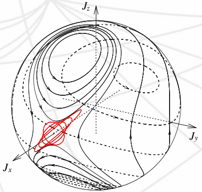
- spin squeezing,
- BEC self trapping,
- excited state quantum phase transitions, etc.

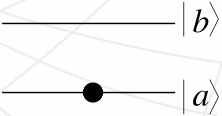
Conclusion

"I went on to work out equations of wobbles. Then I thought about how electron orbits start to move in relativity. Then there's the Dirac Equation in electrodynamics. And then quantum electrodynamics. [...] It was effortless. It was easy to play with these things. It was like uncorking a bottle: Everything flowed out effortlessly. I almost tried to resist it! There was no importance to what I was doing, but ultimately there was. The diagrams and the whole business that I got the Nobel Prize for came from that piddling around with the wobbling plate."

"Surely, You Are Joking, Mr. Feynman!" (Norton, New York, 1985).

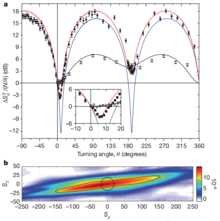




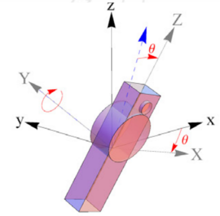


Thanks for your attention

GPS stabilized by momentum wheels



Riedel et al, Nature 2010



Bharadwaj et al, The diver with

a rotor, Ind. Math. 2016