# Bilattices in computer science

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# Bilattices, related structures, and computer science applications

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# Outline

1. Motivation
2. Bilattices
   - Mathematics
   - Logic
3. Residuated multilattices
   - Homomorphisms
   - Congruences
   - Filters
4. Future work

# Why bilattices? (in a nutshell)

A bilattice is a doubly-ordered set satisfying certain properties which has applications in the following lines:

- Kripke-style truth-revision theory (underlying structure)
- Modal logic (modal operators parameterised by bilattice)
- Algebraic study of bilattice-based logics
- Logic programming (many-valuedness, wfs)
- Semantics of natural language questions
- Open constraint programming
- Uncertainty modelling
- Fuzzy logic

# Some historical remarks

- Many inference systems in AI can be unified within a many-valued framework whose truth-values space is a *bilattice*
- Bilattices were proposed by Ginsberg in 1988 as a uniform framework for inference in AI which, besides the order associated with the degree of truth, contains another natural ordering
- Later, Fitting considered applications to LP, to philosophical problems such as the theory of truth, and studied their relationship with a family of many-valued systems generalizing Kleene's three-valued logics
- Other applications include the analysis of entailment, implicature and presupposition in natural language, the semantics of natural language questions, epistemic logic, . . .

# More historical remarks

- In the nineties, bilattices were also investigated by Arieli and Avron, both from an algebraic and from a logical point of view
- Logical bilattices were used to to deal with paraconsistency and non-monotonic reasoning in AI
- Lallouet gave a framework for Bilattice-valued Constraint Programming which allows to represent incomplete/conflicting information and to combine constraints with a set of operators
- Cornelis *et al* used bilattices as a framework for representing uncertain and potentially conflicting information, as in $L$-fuzzy set theory
- Rivieccio made an algebraic study of bilattice-based logics

# Different approaches to bilattices

- From Mathematics
  - Algebraic structure and properties
  - General constructions
- From Logic
  - Semantics of proof systems
  - Paraconsistency
- From Computer Science
  - Generalized logic programming
  - Uncertainty modeling

# Outline

# Bilattices

## and related structures

### Definitions

- A **prebilattice** is a tuple $(B, \leq_t, \leq_k)$, such that both $(B, \leq_t)$ and $(B, \leq_k)$ are bounded lattices
- A **bilattice** is a tuple $(B, \leq_t, \leq_k, \neg)$, such that $(B, \leq_t, \leq_k, \neg)$ is a prebilattice and $\neg$ is an involutive negation wrt $\leq_t$ on $B$.

### Notation

- $t, f$ (resp. $\top, \bot$) are top and bottom elements wrt $\leq_t$ (resp. $\leq_k$)
- $\vee$ and $\wedge$ (resp. $\otimes, \oplus$) are join and meet wrt $\leq_t$ (resp. $\leq_k$)

# Bilattices

## Playing with negations

### Lemma

- $\neg(a \wedge b) = \neg a \vee \neg b \qquad \neg(a \vee b) = \neg a \wedge \neg b$
  $\neg(a \otimes b) = \neg a \otimes \neg b \qquad \neg(a \oplus b) = \neg a \oplus \neg b$
- $\neg f = t, \quad \neg t = f, \quad \neg\bot = \bot, \quad \neg\top = \top$

### Lemma

*If $-$ is a conflation operator (a negation wrt $\leq_k$ on $B$), then*

- $-(a \wedge b) = -a \wedge -b \qquad -(a \vee b) = -a \vee -b$
  $-(a \otimes b) = -a \oplus -b \qquad -(a \oplus b) = -a \otimes -b$
- $-f = f, \quad -t = t, \quad -\bot = \top, \quad -\top = \bot$

# Bilattices

## Additional properties

### Definitions

- A (pre)bilattice $\mathcal{B}$ satisfies the interlacing condition if and only if $\sup_t, \inf_t, \sup_k$ and $\inf_k$ are monotonic with respect to both $\leq_t$ and $\leq_k$. In such a case we say that $\mathcal{B}$ is **interlaced**
- A (pre)bilattice is **distributive** if all the (twelve) possible distributive laws concerning $\wedge, \vee, \otimes, \oplus$ hold

### Theorem (Fitting)

*Every distributive (pre)bilattice is interlaced*

### Lemma

*If $B$ is interlaced, then* $\bot \wedge \top = f, \quad \bot \vee \top = t, \quad f \otimes t = \bot, \quad f \oplus t = \top$

# Constructing bilattices

## The product

### Definition

Given two bounded lattices $(L_1, \leq_1)$ and $(L_2, \leq_2)$, $L_1 \odot L_2$ denotes the pre-bilattice $(L_1 \times L_2, \leq_t, \leq_k)$ where:

- $\langle a, b\rangle \leq_t \langle c, d\rangle$ if and only if $a \leq_1 c$ and $b \geq_2 d$
- $\langle a, b\rangle \leq_k \langle c, d\rangle$ if and only if $a \leq_1 c$ and $b \leq_2 d$

If $L_1 = L_2 = L$ and $\sqcup$ and $\sqcap$ are the join and meet in $L$, then in $L \odot L$:

$(a, b) \vee (c, d) = (a \sqcup c, b \sqcap d)$ $\quad$ $(a, b) \wedge (c, d) = (a \sqcap c, b \sqcup d)$

$(a, b) \oplus (c, d) = (a \sqcup c, b \sqcup d)$ $\quad$ $(a, b) \otimes (c, d) = (a \sqcap c, b \sqcap d)$

$\neg(a, b) = (b, a)$

Furthermore, if 1 and 0 are the top and bottom elements of $L$, then

$$\bot = (0, 0), \quad \top = (1, 1), \quad t = (1, 0), \quad f = (0, 1)$$

# Constructing bilattices

## The product

### Theorem

- *$L \odot L$ is always interlaced*
- *$L \odot L$ is distributive if so is $L$*
- *Every distributive bilattice is isomorphic to $L \odot L$ for some complete distributive lattice*
- *Every interlaced bilattice is isomorphic to $L \odot L$ for some complete lattice*

*If $(x, y) \in L \odot L$, then $x$ represents the information* **for** *some assertion, and $y$ is the information* **against** *it*

# Constructing bilattices

## The intervals

### Definition

Given a complete lattice $(L, \leq_L)$, we define $(\mathcal{I}(L), \leq_t, \leq_k)$ by:

- $\mathcal{I}(L) = \{[a, b,] \mid a \leq_L b\}$
- $[a, b] \leq_t [c, d]$ if and only if $a \leq_L c$ and $b \leq_L d$
- $[a, b] \leq_k [c, d]$ if and only if $a \leq_L c$ and $b \geq_L d$

The intuition is that intervals represent uncertain measures; $\leq_t$ compares degree of truth by ‘shifting rightwards’; $\leq_k$ compares approximations by ‘interval narrowing’

Note the similarity with the product construction.

# Constructing bilattices

## The intervals

Let $(L, \leq_L)$ be a complete lattice with an involutive negation $-$

- A conflation operator can be defined on $L \odot L$ by $-(a, b) = (-b, -a)$
- An element $(a, b) \in L \odot L$ is **coherent** if $(a, b) \leq_k -(a, b)$

**Theorem**

*$\mathcal{I}(L)$ is isomorphic to the substructure of the coherent elements of $L \odot L$*

# Abstract approach to logic

**Definition**

A consequence relation in a language $L$ is a relation $\vdash$ between $2^L$ and $L$ satisfying

- Reflexivity $\varphi \vdash \varphi$
- Monotonicity If $\Gamma \vdash \varphi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash \varphi$
- Transitivity If $\Gamma \vdash \varphi$ and $\Gamma, \varphi \vdash \psi$, then $\Gamma \cup \Gamma' \vdash \psi$

**Definition**

A **propositional logic** is a pair $\langle L, \vdash \rangle$ where $L$ is a *propositional* language and $\vdash$ is a *consequence relation* for $L$.

# Matrices and semantics

## Definitions

A **matrix** for $L$ us a triple $\mathcal{M} = (\mathcal{V}, \mathcal{D}, \mathcal{O})$ where

- $\mathcal{V}$ is the set of *truth-values*
- $\mathcal{D}$ is the set of *designated elements* of $\mathcal{V}$
- $\mathcal{O}$ are the *truth-tables* (interpretations) of the connectives in $L$

A matrix allows for defining the standard semantic notions of valuation and model.

$\Gamma \vdash_{\mathcal{M}} \varphi$ iff $mod_{\mathcal{M}}(\Gamma) \subseteq mod_{\mathcal{M}}(\varphi)$

## Theorem

*The relation $\vdash_{\mathcal{M}}$ is a consequence relation on $L$ and, hence, $\langle L, \vdash_{\mathcal{M}} \rangle$ is a propositional logic (induced by $\mathcal{M}$)*

# Logics via bilattices

Matrices defined on bilattices generate interesting logics because:

- It is possible to incorporate information-based considerations.
- There are ways of representing different levels of inconsistency and incompleteness.

When defining a bilattice-based logic:

- The interpretations of the connectives are usually defined by the basic $\leq_t$-ordering
- The choice of the designated elements in a multiple-valued setting is usually done as a filter or, even, a prime (ultra-)filter in $\mathcal{V}$
- Dual notions for lattice filters and prime-filters are needed

# Logical bilattices

## Bifilters

### Definition

Let $(B, \leq_t, \leq_k, \neg)$ be a bilattice

1. A **bifilter** of $B$ is a nonempty subset $F \subseteq B$ such that
   1. $a \wedge b \in F$ iff $a \in F$ and $b \in F$
   2. $a \otimes b \in F$ iff $a \in F$ and $b \in F$
2. A bifilter $F$ is **prime** if the following holds:
   1. $a \vee b \in F$ iff $a \in F$ or $b \in F$
   2. $a \oplus b \in F$ iff $a \in F$ or $b \in F$

# Logical bilattices

**Definition**

A **logical bilattice** is a pair $(B, F)$ where $B$ is a bilattice and $F$ is a prime bifilter of $B$

- Logical bilattices can be used for defining logics similarly to the way Boolean algebras and prime filters are used.

**Theorem**

*Let* $(B, F)$ *be a logical bilattice. There exists a unique homomorphism* $h\colon B \to \mathcal{FOUR}$ *such that* $h(b) \in \{t, \top\}$ *if and only if* $b \in F$

- Every complete distributive lattice can be turned into a logical bilattice
- Every distributive bilattice can be turned into a logical bilattice

# Logical bilattices

## Definition

Let $B$ be a bilattice. Consider

- $D_k(B) = \{x \mid x \geq_k t\}$ (designated values of $B$ wrt $\leq_k$)
- $D_t(B) = \{x \mid x \geq_t \top\}$ (designated values of $B$ wrt $\leq_t$)

$D_k(B)$ seems to be a particularly natural candidate to play the role of the designated values of $B$

## Lemma

- $t, \top \in D_k(B) \cap D_t(B)$
- $f, \perp \notin D_k(B) \cup D_t(B)$
- $D_k(B) \cup D_t(B)$ *is included in any bifilter*
- *If* $D_k(B) = D_t(B)$, *and this holds for interlaced bilattices, then this is the smaller bifilter*

# Residuated bilattices

- Recently, Jansana and Rivieccio introduced a new product bilattice construction, allowing to obtain a bilattice with two residuated pairs as a certain kind of power of a biresiduated lattice
- Actually, they use a slightly more general notion of biresiduated lattice, in that the existence of a unit element for the product is not required

# Residuated bilattices

## The construction

### Definitions

- Given a biresiduated lattice $(L, \sqcap, \sqcup, \cdot, \backslash, /)$ the **product biresiduated bilattice** is defined as $L \odot L$ together with the operations $\supset, \subset$ given, for all $(a_1, a_2), (b_1, b_2) \in L \times L$, by

$$(a_1, a_2) \supset (b_1, b_2) = (a_1 \backslash b_1, b_2 \cdot a_1)$$
$$(a_1, a_2) \subset (b_1, b_2) = (a_1 / b_1, b_1 \cdot a_2)$$

- The following derived operations are crucial. For all $\alpha, \beta \in L \times L$

$$\alpha \to \beta = (\alpha \supset \beta) \land (\neg\alpha \subset \neg\beta)$$
$$\alpha \leftarrow \beta = \neg\alpha \to \neg\beta$$
$$\alpha * \beta = \neg(\beta \to \neg\alpha)$$

# Residuated bilattices

Any product residuated bilattice contains indeed two residuated pairs

**Theorem**

*Let $L \odot L$ be a product biresiduated bilattice. Then, for all $\alpha, \beta, \gamma \in L \times L$,*

$$\alpha * \beta \leq_t \gamma \quad \textit{iff} \quad \beta \leq_t \alpha \rightarrow \gamma \quad \textit{iff} \quad \alpha \leq_t \gamma \leftarrow \beta$$

Bou and Rivieccio proved that the lattice of congruences of any interlaced bilattice $L \odot L$ is isomorphic to that of $L$. The same holds in this more general context.

# Outline

# Multilattices

- Multisupremum (Multisup): minimal element of the set of upper bounds.

### Definition

A poset, $(M, \leq)$, is a **join-multisemilattice** if, for all $a, b, x \in M$,

$$a \leq x \text{ and } b \leq x \text{ implies that there exists } z \in \text{Multisup}(\{a, b\}) \text{ such that } z \leq x$$

Dual property defines the concept of **meet-multisemilatice**.

### Definition

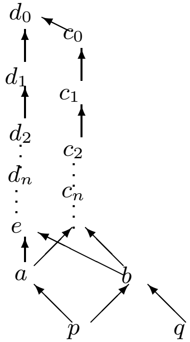
A **multilattice** is a poset $(M, \leq)$ which is a meet and a join-multisemilattice.

# Full multisemilattices and multilattices

## Definition

- A join-multisemilattice $(M, \leq)$ is **full** if, for all $a, b \in M$, Multisup$(\{a, b\}) \neq \varnothing$.
- A meet-multisemilattice $(M, \leq)$ is **full** if, for all $a, b \in M$, Multiinf$(\{a, b\}) \neq \varnothing$.
- A multilattice is full if both multisemilattices are full.

# Examples




- Any finite poset is a multilattice.
- The poset in the figure is a meet-multisemilattice but not a join-multisemilattice.
- The set of words in a universal language with the subword relation is a multilattice.

# POCRIMS

### Definition

A tuple $(A, *, \to, 1, \leq)$ is said to be a *partially ordered commutative residuated integral monoid*, briefly a **pocrim**, if the following properties hold:

- $(A, *, 1)$ is a commutative monoid.
- $(A, \leq)$ is a partially ordered set in which $1$ is the maximum.
- The *residuum property* holds. That is, for every $a, b, c \in A$,

$$a * b \leq c \quad \text{if and only if} \quad a \leq b \to c$$

# Residuated multilattices

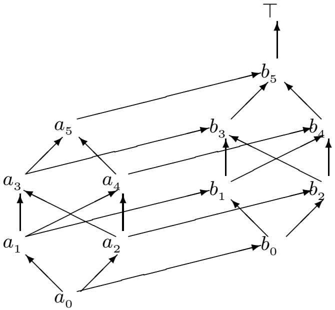
## Definition

A tuple $(M, \sqcup, \sqcap, *, \rightarrow, 1)$ is said to be a **residuated multilattice** if the following properties hold:

- $(M, *, 1)$ is a commutative monoid.
- $(M, \sqcup, \sqcap)$ is a multilattice in which $1$ is the maximum.
- The *residuum property* holds.

# Example

Let $A = \{a_i \mid 0 \leq i \leq 5\}, B = \{b_i \mid 0 \leq i \leq 5\}$ and $C = \{b_2, b_3, b_4, b_5\}$




$$x * y = \begin{cases} x & \text{if } y = \top \\ y & \text{if } x = \top \\ b_2 & \text{if } x, y \in C \\ b_0 & \text{if } x \in B \smallsetminus C, y \in B \\ b_0 & \text{if } x \in B, y \in B \smallsetminus C \\ a_0 & \text{otherwise.} \end{cases}$$

$$x \to y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{if } x = \top \\ a_5 & \text{if } x \in B, y \in A \\ b_1 & \text{if } x \in C \\ & \text{and } y \in B \smallsetminus C \\ b_5 & \text{otherwise.} \end{cases}$$

# Homomorphisms between residuated multilattices

## Definition

Let $h\colon M \to M'$ be a map between residuated multilattices, $h$ is said to be a **homomorphism** if, for all $a, b \in M$,

- $h(a * b) = h(a) * h(b)$, $h(a \to b) = h(a) \to h(b)$
- $h(a \sqcup b) \subseteq h(a) \sqcup h(b)$, and $h(a \sqcap b) \subseteq h(a) \sqcap h(b)$

As a consequence the following conditions hold:

- $h(1) = 1$
- $h(a \sqcup b) = \big(h(a) \sqcup h(b)\big) \cap h(M)$ and
  $h(a \sqcap b) = \big(h(a) \sqcap h(b)\big) \cap h(M)$

# Congruences on residuated multilattices

## Definition

A **congruence** on $M$ is any equivalence relation $\equiv$ such that, for all $a, b, c \in M$, if $a \equiv b$, then

- $a * c \equiv b * c$, $a \to c \equiv b \to c$, $c \to a \equiv c \to b$,
- $a \sqcup c \mathrel{\widehat{\equiv}} b \sqcup c$, and $a \sqcap c \mathrel{\widehat{\equiv}} b \sqcap c$,

## Theorem

*Given a homomorphism $h\colon M \to M'$, the* **kernel relation**, *defined as $a \equiv b$ if and only if $h(a) = h(b)$, is a congruence.*

## Theorem

*If $\equiv$ a congruence on $M$, then the mapping $p\colon M \to M/\equiv$ such that $p(x) = [x]$ is a surjective homomorphism.*

# Filters in pocrims

**Definition**

Given $\mathcal{A} = (A, \leq, *, \rightarrow, 1)$ a pocrim, $\varnothing \neq F \subseteq A$ is said to be a **filter** if the following conditions hold:

i) if $a, b \in F$, then $a * b \in F$

ii) if $a \leq b$ and $a \in F$, then $b \in F$.

On the other hand, $F$ is said to be a **deductive system** if

i) $1 \in F$ and

ii) $a \rightarrow b \in F$ and $a \in F$ imply $b \in F$.

Both definitions are equivalent.

# Filters in multilattices

## Definition

Let $(M, \sqcup, \sqcap)$ be a multilattice. A non-empty set $F \subseteq M$ is said to be a **filter** if the following conditions hold:

1. $i, j \in F$ implies $\varnothing \neq i \sqcap j \subseteq F$.
2. $i \in F$ implies $i \sqcup a \subseteq F$ for all $a \in M$.
3. For all $a, b \in M$, if $(a \sqcup b) \cap F \neq \varnothing$ then $a \sqcup b \subseteq F$.

# Filters in residuated multilattices

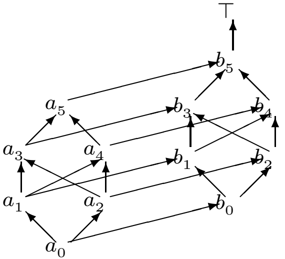
## Definition

A non-empty set in a residuated multilattice is going to be called

- **deductive system** if it is a filter in the underlying pocrim
- **m-filter** if it is a filter in the underlying multilattice
- **r-filter** if it is both a deductive system and an $m$-filter

# Example

Let $A = \{a_i \mid 0 \leq i \leq 5\}$, $B = \{b_i \mid 0 \leq i \leq 5\}$ and $C = \{b_i \mid 2 \leq i \leq 5\}$




- $C \cup \{\top\}$ is a deductive system but it is not an $m$-filter because $b_3 \sqcap b_4 = \{b_1, b_2\} \not\subseteq C$.
- $\{b_5, \top\}$ is an $m$-filter that is not a deductive system because $b_5 * b_5 = b_2 \notin \{b_5, \top\}$.
- $B \cup \{\top\}$ is both a deductive system and an $m$-filter.

# Filters in residuated multilattices

**Theorem**

*Let $h\colon M \to M'$ be a homomorphism between residuated multilattices. Then $h^{-1}(1) = \{x \in M \mid h(x) = 1\}$ is an $r$-filter of $M$, the* **kernel filter**.

**Theorem**

*Let $M$ be a residuated multilattice, $1$ the top element and $\equiv$ a congruence. The equivalence class $[1]$ is an $r$-filter.*

# Outline

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3. Residuated multilattices
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   - Filters

4. Future work

# Future work

- Kind of filters that defines a congruence in a residuated multilattice
- Algebraic properties in a residuated multilattice
- Considering the residuated operations $\odot$ and $\rightarrow$ as hyperoperations, thus leading to a complete embedding of the structure into a hyperalgebraic framework.

# Bilattices, related structures, and computer science applications

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