Removing an incidence from a formal context

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Motivation

- Computing a concept lattice is computationally hard.
- Slight change in input data $\rightarrow$ recomputing the concept lattice.
- Better to use an incremental algorithm.
- Several incremental algorithms available.
- Most of them work at object/attribute level.
- We are trying different approach – working at incidence level.

Why removing an incidence?

- In a sense, it can be considered a basic problem.
- A “full” row in a table – no change in the structure of the concept lattice.
- Remove unwanted incidences afterwards.
- Start with a “full” context and by removing incidences acquire any context.
- Analysis of structural differences after the smallest change is interesting from theoretical point of view.
- Might lead to an efficient way of computing lattices of two similar contexts.

Problem statement

- Let $\langle X, Y, I\rangle$ (*old*) and $\langle X, Y, J\rangle$ (*new*) be formal contexts such that:
  - J results from I by removing exactly one incidence;
  - the removed incidence corresponds to object x_0 and attribute y_0.
- Denote respective concept lattices $\mathcal{B}(I)$ (*old*) and $\mathcal{B}(J)$ (*new*).

	$\boldsymbol{y_0}$	y_1	y_2
$\boldsymbol{x_0}$	×	×	
x_1	×	×	
x_2			×

	$\boldsymbol{y_0}$	y_1	y_2
$\boldsymbol{x_0}$		×	
x_1	×	×	
x_2			×

Question: Can we transform $\mathcal{B}(I)$ to $\mathcal{B}(J)$ without computing $\mathcal{B}(J)$ from scratch?

What is in the paper?

- Analysis of concept changes and structural changes upon removal of an incidence.
- Two basic algorithms based on this analysis.

We use the following four operators:
For concepts $c = \langle A, B\rangle \in \mathcal{B}(I), d = \langle C, D\rangle \in \mathcal{B}(J)$ we set

$$c^{\square} = \langle A^{\square}, B^{\square}\rangle = \langle A^{\uparrow_J\downarrow_J}, A^{\uparrow_J}\rangle, \qquad c_{\square} = \langle A_{\square}, B_{\square}\rangle = \langle B^{\downarrow_J}, B^{\downarrow_J\uparrow_J}\rangle,$$
$$d^{\boxtimes} = \langle C^{\boxtimes}, D^{\boxtimes}\rangle = \langle D^{\downarrow_I}, D^{\downarrow_I\uparrow_I}\rangle, \qquad d_{\boxtimes} = \langle C_{\boxtimes}, D_{\boxtimes}\rangle = \langle C^{\uparrow_I\downarrow_I}, C^{\uparrow_I}\rangle.$$

- Obviously $c^{\square}, c_{\square} \in \mathcal{B}(J)$ and $d^{\boxtimes}, d_{\boxtimes} \in \mathcal{B}(I)$. $c^{\square}$ (resp. $c_{\square}$) is called *the upper* (resp. *lower*) *child of* c in $\mathcal{B}(J)$.
- In this case, $d^{\boxtimes} = d_{\boxtimes}$ and it is the (unique) concept from $\mathcal{B}(I)$, containing, as a rectangle, the rectangle represented by d.

Children

Proposition (child operators)

The mappings $c \mapsto c^{\Box}$, $c \mapsto c_{\Box}$, *and* $d \mapsto d^{\boxtimes}, d \mapsto d_{\boxtimes}$ *are isotone and satisfy*

$$c \le c^{\Box\boxtimes}, \qquad d \le d^{\boxtimes\Box}, \qquad c^{\Box\boxtimes\Box} = c^{\Box}, \qquad d^{\boxtimes\Box\boxtimes} = d^{\boxtimes},$$
$$c \ge c_{\Box\boxtimes}, \qquad d \ge d_{\boxtimes\Box}, \qquad c_{\Box\boxtimes\Box} = c_{\Box}, \qquad d_{\boxtimes\Box\boxtimes} = d_{\boxtimes}.$$

- Shows basic properties of proposed operators.
- Compositions $^{\Box\boxtimes}$, $^{\boxtimes\Box}$ are closure operators.
- Compositions $_{\Box\boxtimes}$, $_{\boxtimes\Box}$ are interior operators.

Stable concepts

Definition (stable concepts)

Formal concepts from the intersection $\mathcal{B}(I) \cap \mathcal{B}(J)$ are called *stable*.

- Not influenced by removal of the incidence $\langle x_0, y_0 \rangle$ from I.
- Do not have to be recomputed.
- A set of their neighbors might change.

Non-stable concepts

- Non-stable concepts play major role.
- By identifying non-stable concepts we also identify stable ones.
- They form an interval in $\mathcal{B}(I)$.

Proposition

A concept $c \in \mathcal{B}(I)$ *is not stable iff* $c \in [\gamma_I(x_0), \mu_I(y_0)]$.

Characterization of stable concepts

Proposition (stable concepts in $\mathcal{B}(I)$)

The following assertions are equivalent for a concept $c \in \mathcal{B}(I)$:

1. c *is stable,*
2. $c \notin [\gamma_I(x_0), \mu_I(y_0)]$*,*
3. $c = c^{\square}$*,*
4. $c = c_{\square}$*,*
5. $c^{\square} = c_{\square}$*.*

Proposition (stable concepts in $\mathcal{B}(J)$)

The following assertions are equivalent for a concept $d \in \mathcal{B}(J)$:

1. d *is stable,*
2. $d = d^{\boxtimes}$*,*
3. $d^{\boxtimes}$ *is stable.*

The role of child concepts

Recall that for $c \in \mathcal{B}(I)$ $c^{\Box}$ (resp. $c_{\Box}$) is called *the upper* (resp. *lower*) *child of* c in $\mathcal{B}(J)$.

Proposition

A non-stable concept $d \in \mathcal{B}(J)$ is a (upper or lower) child of exactly one concept $c \in \mathcal{B}(I)$. This concept is non-stable and satisfies $c = d^{\boxtimes} = d_{\boxtimes}$.

- Shows importance of child concepts.
- Relation between non-stable concepts form $\mathcal{B}(J)$ and $\mathcal{B}(I)$.

Transforming concepts

The previous leads to the following simple way of constructing $\mathcal{B}(J)$ from $\mathcal{B}(I)$.

For each $c \in \mathcal{B}(I)$ the following has to be done:

1. If c is stable, then it has to be added to $\mathcal{B}(J)$.
2. If c is not stable, then each its non-stable child (i.e., each non-stable element of $\{c^{\square}, c_{\square}\}$) has to be added to $\mathcal{B}(J)$.

- All proper concepts are added exactly once.
- None will be omitted.

Transforming the structure

- Transforming concepts has been shown to be easy.
- Transforming the structure of concept lattice is more difficult.
- For this, more insight into structural properties of ${}^{\Box\boxtimes}$, ${}_{\Box\boxtimes}$ is required.
- The main role of operators ${}^{\Box\boxtimes}$, ${}_{\Box\boxtimes}$.

Proposition

Each stable concept is a fixpoint of both ${}^{\Box\boxtimes}$ *and* ${}_{\Box\boxtimes}$.

Transforming the structure

Recall the following:

- $_{\square\boxtimes}$ is an interior operator on $\mathcal{B}(I)$;
- $^{\square\boxtimes}$ is a closure operator on $\mathcal{B}(I)$;
- also $c \in \mathcal{B}(I)$, $c_{\square\boxtimes} \leq c \leq c^{\square\boxtimes}$;

Thus, we can consider the interval $[c_{\square\boxtimes}, c^{\square\boxtimes}] \subseteq \mathcal{B}(I)$.

Proposition

For any $c \in \mathcal{B}(I)$, each concept from $[c_{\square\boxtimes}, c^{\square\boxtimes}] \setminus \{c\}$ is stable.

Transforming the structure

Another important structural property:

Proposition

Let $c \in \mathcal{B}(I)$ be a non-stable concept. If c is a fixpoint of $^{\square\boxtimes}$, then each $c' \leq c$ is also a fixpoint of $^{\square\boxtimes}$. If c is a fixpoint of $_{\square\boxtimes}$, then each $c' \geq c$ is also a fixpoint of $_{\square\boxtimes}$.

- Premise of non-stability is necessary.
- Gives us restrictions on possible neighbors.

Selected consequences

- Using previous we obtain restriction of neighborhood relationship w.r.t. proposed operators.

Table: Possible neighborhood relationship in $\mathcal{B}(I)$.

concept / neighbors	$c' = c'^{\square\boxtimes}$, $c' = c'_{\square\boxtimes}$	$c' \neq c'^{\square\boxtimes}$, $c' = c'_{\square\boxtimes}$	$c' = c'^{\square\boxtimes}$, $c' \neq c'_{\square\boxtimes}$	$c' \neq c'^{\square\boxtimes}$, $c' \neq c'_{\square\boxtimes}$
$c = c^{\square\boxtimes}, c = c_{\square\boxtimes}$	$\nearrow\swarrow$	$\nearrow$	$\swarrow$	
$c \neq c^{\square\boxtimes}, c = c_{\square\boxtimes}$	$\swarrow$	$\nearrow\swarrow$	$\swarrow$	$\swarrow$
$c = c^{\square\boxtimes}, c \neq c_{\square\boxtimes}$	$\nearrow$	$\nearrow$	$\nearrow\swarrow$	$\nearrow$
$c \neq c^{\square\boxtimes}, c \neq c_{\square\boxtimes}$		$\nearrow$	$\swarrow$	$\nearrow\swarrow$

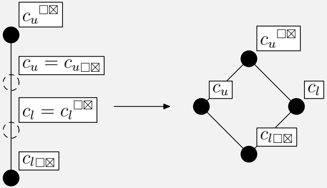
Structure of the transformation algorithm

Algorithm 1 Transforming $\mathcal{B}(I)$ with structural information into $\mathcal{B}(J)$.

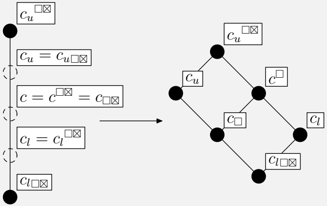
procedure TRANSFORMCONCEPTLATTICE($\mathcal{B}(I)$)
  for all $c = \langle A, B\rangle \in [\gamma_I(x_0), \mu_I(y_0)]$ from least to largest w.r.t. $\sqsubseteq$ **do**
    if $c = c^{\Box\boxtimes}$ **and** $c = c_{\Box\boxtimes}$ **then** ▷ Concept will split.
      $\mathcal{B}(I) \leftarrow \mathcal{B}(I) \setminus \{c\}$;
      $\mathcal{B}(I) \leftarrow \mathcal{B}(I) \cup SplitConcept(c)$;
    else if $c \neq c^{\Box\boxtimes}$ **and** $c = c_{\Box\boxtimes}$ **then** ▷ Extent will be smaller.
      $A \leftarrow A \setminus \{x_0\}$;
    else if $c = c^{\Box\boxtimes}$ **and** $c \neq c_{\Box\boxtimes}$ **then** ▷ Intent will be smaller.
      $RelinkReducedIntent(c)$;
      $B \leftarrow B \setminus \{y_0\}$;
    else if $c \neq c^{\Box\boxtimes}$ **and** $c \neq c_{\Box\boxtimes}$ **then** ▷ Concept will vanish.
      $\mathcal{B}(I) \leftarrow \mathcal{B}(I) \setminus \{c\}$;
      $UnlinkVanishedConcept(c)$;
    end if
  end for
end procedure

Example

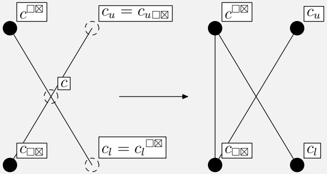
Examples of transformations of non-stable concepts from $\mathcal{B}(I)$ into concepts of $\mathcal{B}(J)$.



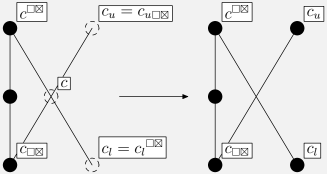
(a) Concepts become incomparable.



(b) Concept in middle "splits into two".



(c) Concept in the middle vanishes.



(d) Concept in the middle vanishes.

Consider following contexts:

	y_1	y_2	y_3	y_4	$\boldsymbol{y_0}$
$\boldsymbol{x_0}$		×		×	×
x_1			×	×	×
x_2				×	
x_3	×	×			×
x_4		×			

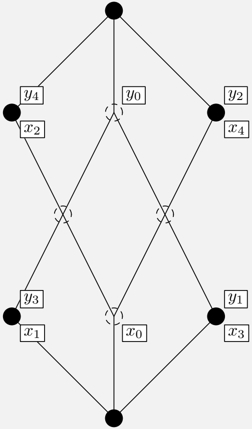
(a) The old context $\langle X, Y, I \rangle$.

	y_1	y_2	y_3	y_4	$\boldsymbol{y_0}$
$\boldsymbol{x_0}$		×		×	
x_1			×	×	×
x_2				×	
x_3	×	×			×
x_4		×			

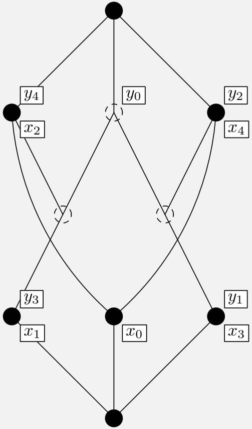
(b) The new context $\langle X, Y, J \rangle$.

Example

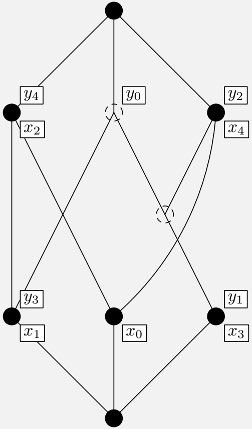
Each picture captures the state of the lattice after transformation of a non-stable concept. Non-stable concepts are drawn with dashed lines.



(a) Initial state of $\mathcal{B}(I)$.

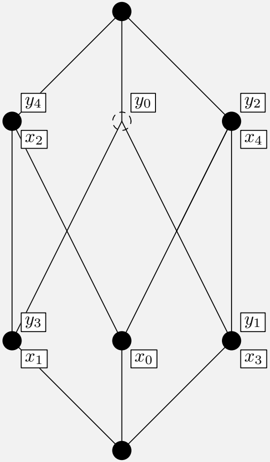


(b) Transformation of concept $\gamma_I(x_0)$.

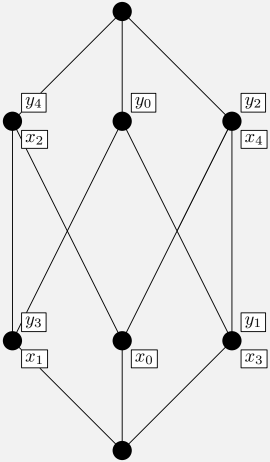


(c) One of the middle concepts vanishes.

Example



(d) Remaining middle concept vanishes.



(e) Transformation of the last non-stable concept $\mu_I(y_0)$.

Conclusion

- We presented analysis of possible structural changes in a concept lattice upon removal of an incidence.
- Two algorithms based on this analysis.
- Proposed algorithms could be further optimized.
- We have a few results that could be used for optimization of proposed algorithms.
- Preliminary experiments show that usually there is considerably less non-stable concepts than stable ones - cutting running time of proposed algorithms.

Remarks

- Updating a concept lattice after removal of an incidence seems to be, in a sense, easier.
- Removing an incidence seems more natural then adding it.
- By removing incidences we can generate all concept lattices.
- It came to our attention that R. Wille coined this "killing a cross" in the early days of FCA and considered it a important step to solve some theoretical problems.