Formal L-concepts with Rough Intents

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Motivation and Aims

Motivation

- Many models involving ordinary sets and relations have been subject to extensions in which the ordinary sets and relations are replaced by fuzzy sets and fuzzy relations.
- While the natural reason for such extensions comes from the need to extend the applicability of the models, the technical side of the extensions is far from being obvious.
- Various methods have been proposed (the best-known concept of representation of fuzzy sets by cuts.)
- We focus on models based on closure-like structures derived from a binary relation.

Aim of the paper

- we provide a simple proof of the so-called basic theorem of a general type of concept lattices and generalize several existing approaches to concept lattices.
- we promote a useful representation of fuzzy sets – Cartesian representation.

Outline

- Brief Introduction to Formal Concept Analysis
- Generalized Concept Lattices and the Main Theorem
- The Cartesian Representation
- The Simple Proof of the Main Theorem
- Summary

Formal Concept Analysis

(Wille, Germany, 1982) non-numerical method for identification of formal concepts (based on logic/algebra/discrete math)

INPUT: Context

	y_1	y_2	y_3	y_4
x_1	1	1	1	1
x_2	1	0	1	1
x_3	0	1	1	1
x_4	0	1	1	1
x_5	1	0	1	0

$X = \{x_1, x_2 \dots\}$ ... objects (rows)

$Y = \{y_1, y_2 \dots\}$ ... attributes (columns)

I ... relation of incidence

$\langle x, y \rangle \in I =$ (1 in the table) ...

... object has attribute

OUTPUTS:

- Concept lattices
- Attribute implications

Formal Concept Analysis

Concept lattices

Induced operators . . . mappings $^{\uparrow}$, $^{\downarrow}$.
$A \subseteq X \mapsto A^{\uparrow}$ . . . attributes common to all objects from A
$B \subseteq Y \mapsto B^{\downarrow}$ . . . objects sharing all attributes from B
Formal Concept in $\langle X, Y, I \rangle$ . . . $\langle A, B \rangle$, $A \subseteq X$, $B \subseteq Y$, s.t.

$$A^{\uparrow} = B \text{ and } B^{\downarrow} = A$$

A . . . extent . . . objects covered by formal concept
B . . . intent . . . attributes covered by formal concept

Example: DOG (extent = collection of all dogs (foxhound, poodle, . . .), intent = {barks, has four limbs, has tail, . . . })

Subconcept–superconcept ordering $\leq$ of formal concept is defined by

$$\langle A_1, B_1 \rangle \leq \langle A_2, B_2 \rangle \text{ iff } A_1 \subseteq A_2 (\text{iff } B_2 \subseteq B_1)$$

Example: DOG $\leq$ ANIMAL $\leq$ ORGANISM

Formal Concept Analysis

Concept Lattice and the Main Theorem

$\mathcal{B}(X,Y,I) = \{\langle A,B\rangle \mid A^{\uparrow} = B, B^{\downarrow} = A\} + \leq$ is called a concept lattice.

Theorem

(1) The set $\mathcal{B}(X.Y,I) = \{\langle A,B\rangle \mid A^{\uparrow} = B, B^{\downarrow} = A\}$ *with* $\leq$ *infima and suprema defined as follows*

$$\bigwedge_{j\in J} \langle A_j, B_j\rangle = \langle \bigcap_{j\in J} A_j, (\bigcup_{j\in J} B_j)^{\downarrow\uparrow}\rangle \qquad \bigvee_{j\in J} \langle A_j, B_j\rangle = \langle (\bigcup_{j\in J} A_j)^{\uparrow\downarrow}, \bigcap_{j\in J} B_j, \rangle$$

(2) Moreover, a complete lattice $\mathbf{V} = \langle V, \leq\rangle$ *is isomorphic to* $\mathcal{B}(X,Y,I)$ *iff there are mappings* $\gamma : X \to V$ *and* $\mu : Y \to V$ *such that* $\gamma(X)$ *is supremally dense in* $\mathbf{V}$*,* $\mu(Y)$ *is infimally dense in* $\mathbf{V}$*, and* $\langle x,y\rangle \in I$ *is equivalent to* $\gamma(x) \leq \mu(y)$ *for all* $x \in X, y \in Y$*.*

Generalization of FCA

Instead of

	y_1	y_2	y_3	y_4
x_1	1	1	1	1
x_2	1	0	1	1
x_3	0	1	1	1
x_4	0	1	1	1
x_5	1	0	1	0

$X = \{x_1, x_2 \dots\}$ ... objects (rows)

$Y = \{y_1, y_2 \dots\}$ ... attributes (columns)

I ... relation of incidence

$\langle x, y \rangle \in I$ = (1 in the table) ...

... object has attribute

we have

	y_1	y_2	y_3	y_4
x_1	0.8	1	0.3	1
x_2	0.9	0	1	1
x_3	0	0.1	0.3	1
x_4	0	1	1	1
x_5	1	0	0.7	0

$X = \{x_1, x_2 \dots\}$ ... objects (rows)

$Y = \{y_1, y_2 \dots\}$ ... attributes (columns)

I ... relation of incidence

$I(x, y)$ ... degree in which the object x has the attribute y

Supremum preserving aggregation structures

Aggregation structure:
$\mathbf{L}_1 = \langle L_1, \leq_1 \rangle, \mathbf{L}_2 = \langle L_2, \leq_2 \rangle, \mathbf{L}_3 = \langle L_2, \leq_2 \rangle$ – complete lattices, and $\Box : L_1 \times L_2 \to L_3$.
A quadruple $\langle \mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \Box \rangle$ satisfying

$$(\textstyle\bigvee_{1\, j \in J} a_j) \Box b = \textstyle\bigvee_{3\, j \in J} (a_j \Box b) \qquad a \Box (\textstyle\bigvee_{2\, j' \in J'} b_{j'}) = \textstyle\bigvee_{3\, j' \in J'} (a \Box b_{j'}).$$

is called a *(supremum preserving) aggregation structure*.

Operations of residuation:
$\circ_{\Box} : L_1 \times L_3 \to L_2$ and ${}_{\Box}\circ : L_3 \times L_2 \to L_1$ (adjoints to $\Box$) are defined by

$$a_1 \circ_{\Box} a_3 = \textstyle\bigvee_2 \{a_2 \,|\, a_1 \Box a_2 \leq_3 a_3\},$$
$$a_3 \,{}_{\Box}\!\circ\, a_2 = \textstyle\bigvee_1 \{a_1 \,|\, a_1 \Box a_2 \leq_3 a_3\}.$$

(We put indices in a_1 and the like for mnemonic reasons.
Thus, a_1 indicates that a_1 is taken from L_1 and the like.)

Aggregation Structures – Examples

$\langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ – complete residuated lattice with partial order $\leq$.

- $\langle L, \wedge, \vee, 0, 1\rangle$ – complete lattice,
- $\langle L, \otimes, 1\rangle$ – commutative monoid,
- $a \otimes b \leq c$ iff $a \leq b \rightarrow c$ (adjointness).

Consider $L_i = L$ and $\leq_i$ is either $\leq$ or the dual of $\leq$ (i.e. $\leq_i = \leq$ or $\leq_i = \leq^{-1}$).

(a) Let $\mathbf{L}_1 = \langle L, \leq\rangle$, $\mathbf{L}_2 = \langle L, \leq\rangle$, and $\mathbf{L}_3 = \langle L, \leq\rangle$, let $\square$ be $\otimes$. Then, as is well known, $\square$ commutes with suprema in both arguments. Namely, due to commutativity of $\otimes$, commuting amounts to $a \otimes \bigvee_{j\in J} b_j = \bigvee_{j\in J}(a \otimes b_j)$.
Furthermore,

$$a_1 \circ_{\square} a_3 = \bigvee\{a_2 \,|\, a_1 \otimes a_2 \leq a_3\} = a_1 \rightarrow a_3$$

and, similarly, $a_3 \,{}_{\square}{\circ}\, a_2 = a_2 \rightarrow a_3$.

(b) Let $\mathbf{L}_1 = \langle L, \leq \rangle$, $\mathbf{L}_2 = \langle L, \leq^{-1} \rangle$ and $\mathbf{L}_3 = \langle L, \leq^{-1} \rangle$, let $\square$ be $\rightarrow$. Then $\square$ commutes with suprema in both arguments.

Namely, the conditions for commuting with suprema in this case become

$$(\bigvee_{j \in J} a_j) \rightarrow b = \bigwedge_{j \in J} (a_j \rightarrow b) \text{ and } a \rightarrow (\bigwedge_{j \in J} b_j) = \bigwedge_{j \in J} (a \rightarrow b_j)$$

which are well-known properties of residuated lattices.

In this case, we have

$$a_1 \circ_{\square} a_3 = \bigwedge \{ a_2 \mid a_1 \rightarrow a_2 \geq a_3 \} = a_1 \otimes a_3$$

$$a_3 \; {}_{\square}\!\circ \; a_2 = \bigvee \{ a_1 \mid a_1 \rightarrow a_2 \geq a_3 \} = a_3 \rightarrow a_2.$$

Fuzzy Sets and Fuzzy Contexts

Fuzzy sets

Let $\mathbf{L} = \langle L, \leq \rangle$ be a complete lattice and U be ordinary set (universe).
$\mathbf{L}$-set A in U is a mapping $A : U \to L$.
Operations with $\mathbf{L}$-sets defined component-wise using operations of $\mathbf{L}$
System of all $\mathbf{L}$-sets in U denoted L^U.

Let $\langle \mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \Box \rangle$ be a sup-preserving aggregation structure.
$\mathbf{L}_3$-context – $\langle X, Y, I \rangle$:

- X and Y are non-empty sets of objects and attributes, respectively
- $I : X \times Y \to L_3$ is a binary $\mathbf{L}_3$-relation between X and Y.
  For $x \in X$ and $y \in Y$, the degree $I(x, y)$ is interpreted as the degree to which the object x has the attribute y.

Concept-forming operators $^{\uparrow}: {L_1}^X \to {L_2}^Y$ and $^{\downarrow}: {L_2}^Y \to {L_1}^X$ defined by

$$\begin{array}{rcl} A^{\uparrow}(y) & = & \bigwedge_{2 x \in X}(A(x) \circ_{\Box} I(x,y)) \\ B^{\downarrow}(x) & = & \bigwedge_{1 y \in Y}(I(x,y) \, {}_{\Box}\!\circ \, B(y)) \end{array}$$

for any $A \in {L_1}^X$ and $B \in {L_2}^Y$.

Formal concept – pair $\langle A, B \rangle$ consisting of an $\mathbf{L}_1$-set A in X and an $\mathbf{L}_2$-set B in Y for which $A^{\uparrow} = B$ and $B^{\downarrow} = A$.

$\mathcal{B}(X,Y,I)$ denotes the set of all formal concepts of I, i.e.

$$\mathcal{B}(X,Y,I) = \{\langle A, B \rangle \in {L_1}^X \times {L_2}^Y \mid A^{\uparrow} = B,\ B^{\downarrow} = A\}.$$

Subconcept-superconcept hierarchy $\leq$ of formal concept is defined by

$$\langle A_1, B_1 \rangle \leq \langle A_2, B_2 \rangle \text{ iff } A_1 \subseteq A_2 (\text{iff } B_2 \subseteq B_1)$$

Examples of concept-forming operators

(a) Let $\mathbf{L}_1 = \langle L, \leq \rangle$, $\mathbf{L}_2 = \langle L, \leq \rangle$, and $\mathbf{L}_3 = \langle L, \leq \rangle$, let $\square$ be $\otimes$. Fuzzy sets $A^{\uparrow} \in L^Y$ and $B^{\downarrow} \in L^X$:

$$A^{\uparrow}(y) = \bigwedge_{x \in X} A(x) \to I(x, y)$$

$$B^{\downarrow}(x) = \bigwedge_{y \in Y} B(y) \to I(x, y)$$

(b) Let $\mathbf{L}_1 = \langle L, \leq \rangle$, $\mathbf{L}_2 = \langle L, \leq^{-1} \rangle$, $\mathbf{L}_3 = \langle L, \leq^{-1} \rangle$, let $\square$ be $\to$.
Fuzzy sets $A^{\cap} \in L^Y$ and $B^{\cup} \in L^X$:

$$A^{\cap}(y) = \bigvee_{x \in X} A(x) \otimes I(x, y)$$

$$B^{\cup}(x) = \bigwedge_{y \in Y} I(x, y) \to B(y)$$

Theorem

Let $\langle \mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \Box \rangle$ *be a supremum-preserving aggregation structure and* $\langle X, Y, I \rangle$ *be an* $\mathbf{L}_3$*-context.*
(1) $\mathcal{B}(X, Y, I)$ *equipped with* $\leq$ *is a complete lattice with infima and suprema described as:*

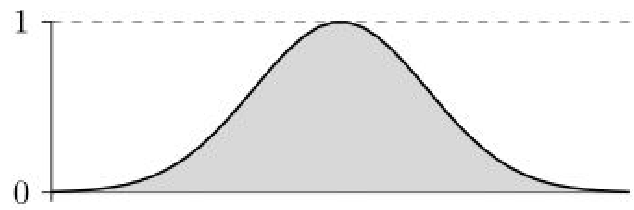
$$\bigwedge_{j \in J} \langle A_j, B_j \rangle = \left\langle \bigcap_{j \in J} A_j, (\bigcup_{j \in J} B_j)^{\uparrow\downarrow} \right\rangle, \bigvee_{j \in J} \langle A_j, B_j \rangle = \left\langle (\bigcup_{j \in J} A_j)^{\downarrow\uparrow}, \bigcap_{j \in J} B_j \right\rangle$$

(2) Moreover, a complete lattice $\mathbf{V} = \langle V, \leq \rangle$ *is isomorphic to* $\mathcal{B}(X, Y, I)$ *iff there are mappings* $\gamma : X \times L_1 \to V$ *and* $\mu : Y \times L_2 \to V$ *such that* $\gamma(X \times L_1)$ *is supremally dense in* $\mathbf{V}$*,* $\mu(Y \times L_2)$ *is infimally dense in* $\mathbf{V}$*, and* $a \Box b \leq_3 I(x, y)$ *is equivalent to* $\gamma(x, a) \leq \mu(y, b)$ *for all* $x \in X, y \in Y, a \in L_1, b \in L_2$*.*

The Cartesian Representation

For a complete lattice $\mathbf{L} = \langle L, \leq \rangle$ and a fuzzy set A in X with truth degrees in L, we put

$$\lfloor A \rfloor = \{\langle x, a \rangle \in X \times L \mid a \leq A(x)\}$$



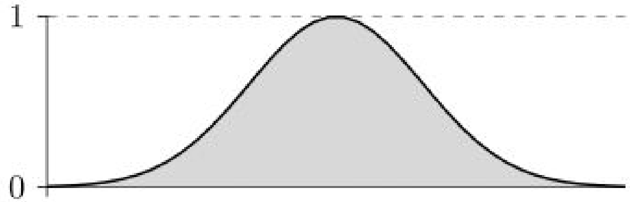
That is, $\lfloor A \rfloor$ is the "area below the membership function".

For an ordinary set $A' \subseteq X \times L$ define an $\mathbf{L}$-set $\lceil A' \rceil$ in X by

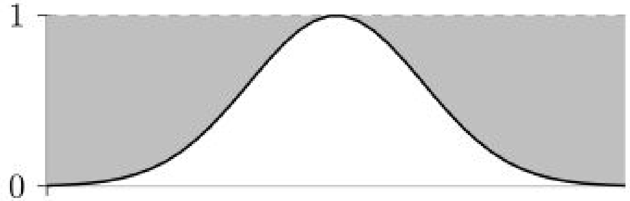
$$\lceil A' \rceil(x) = \bigvee \{a \mid \langle x, a \rangle \in A'\}.$$

Considering aggregation structures from the running examples. . .

(a) $\mathbf{L} = \langle L, \leq \rangle$



(b) $\mathbf{L} = \langle L, \leq^{-1} \rangle$



The Simple Proof

For aggregation structure $\langle \mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \Box \rangle$ and $\mathbf{L}_3$-context $\langle X, Y, I \rangle$, consider the ordinary context $\langle X \times L_1, Y \times L_2, I^{\times} \rangle$, where $I^{\times} \subseteq (X \times L_1) \times (Y \times L_2)$ is defined by

$$\langle \langle x, a \rangle, \langle y, b \rangle \rangle \in I^{\times} \text{ iff } a \,\Box\, b \leq_3 I(x, y).$$

The concept lattice $\mathcal{B}(X, Y, I)$ over $\langle \mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \Box \rangle$ is isomorphic to the ordinary concept lattice $\mathcal{B}(X \times L_1, Y \times L_2, I^{\times})$.

$\varphi \colon \mathcal{B}(X, Y, I) \to \mathcal{B}(X \times L_1, Y \times L_2, I^{\times})$, $\psi \colon \mathcal{B}(X \times L_1, Y \times L_2, I^{\times}) \to \mathcal{B}(X, Y, I)$ defined by

$$\varphi(\langle A, B \rangle) = \langle \lfloor A \rfloor, \lfloor B \rfloor \rangle,$$
$$\psi(\langle A', B' \rangle) = \langle \lceil A' \rceil, \lceil B' \rceil \rangle$$

for $\langle A, B \rangle \in \mathcal{B}(X, Y, I), \langle A', B' \rangle \in \mathcal{B}(X \times L_1, Y \times L_2, I^{\times})$ are well-defined, mutually inverse, order-preserving bijections between the two concept lattices.

Summary

- simple proof of the main theorem for general concept lattices was shown.
- the Cartesian representation is a useful tool in fuzzy set theory and its applications.

To be in the full version of the paper

- alternative proof of the main theorem (using the Cartesian representation)
- more general form of the main theorem (concept-forming parametrized by truth-stressing hedges)

THANK YOU!