Minimal Bases for Temporal Attribute Implications

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Input data for attribute implications

Definition

Formal context is a binary relation $I \subseteq X \times Y$ between a set of objects X and a set of attributes Y.

	a	b	c	d
1	×	×	×	×
2	×			×
3		×	×	
4	×	×	×	×
5			×	

Figure: Formal context

Input data for temporal attribute implications

Definition

Temporal formal context is a ternary relation $I \subseteq X \times Y \times \mathbb{Z}$ where X is a set of objects, Y is a set of attributes, and $\mathbb{Z}$ is a set of time points represented by integers.

…,

$t = 1$

	a	b	c	d
1				×
2	×	×	×	
3	×		×	×
4			×	×
5		×		

,

$t = 2$

	a	b	c	d
1		×	×	×
2		×		×
3	×	×	×	×
4	×		×	
5				×

,

$t = 3$

	a	b	c	d
1			×	×
2	×		×	
3	×	×		×
4		×	×	×
5				

, …

Figure: Series of formal contexts

Temporal attribute implications

Definition

Temporal attribute implication is a formula in the form

$$\left\{y_1^{i_1}, \ldots, y_m^{i_m}\right\} \Rightarrow \left\{z_1^{j_1}, \ldots, z_n^{j_n}\right\},$$

where $y_1, \ldots, y_m, z_1, \ldots, z_n \in Y$ and $i_1, \ldots, i_m, j_1, \ldots, j_n \in \mathbb{Z}$.

- Numbers in temporal attribute implications denote relative time points (0 is current time point, 1 is the follower of 0, etc.).
- Meaning in a series of formal contexts: "Every object in every time point t having attributes y_1 at time point i_1 relative to $t, \ldots, y_m$ at time point i_m relative to t has also attributes z_1 at time point j_1 relative to $t, \ldots, z_n$ at time point j_n relative to t."

Basic notions

- y^i ... attribute y at time point i
- $\mathcal{T}_Y$ ... set of all attributes at all times
- to get absolute time points from relative for $M \subseteq \mathcal{T}_Y$ and $i \in \mathbb{Z}$ we have $M + i = \{y^{t+i} \mid y^t \in M\}$
- objects of temporal formal contexts can be regarded as subsets of $\mathcal{T}_Y$

Definition

Attribute implication annotated by time points $A \Rightarrow B$ *is true in* $M \subseteq \mathcal{T}_Y$ whenever, for each $i \in \mathbb{Z}$, if $A + i \subseteq M$, then $B + i \subseteq M$. This fact is denoted by $M \models A \Rightarrow B$.

Example

Let $M_2 = \{c^1, s^1, c^2, r^2, c^3, s^3\}$. Then $M_2 \models \{c^0, r^0\} \Rightarrow \{c^1\}$ but $M_2 \not\models \{c^0\} \Rightarrow \{s^1\}$ where integer 1 is a counter example.

Entailment

Definition

A set $M \subseteq \mathcal{T}_Y$ is a *model* of a set Σ of attribute implications annotated by time points (called *theory*) if every $A \Rightarrow B$ from Σ is true in M.

Definition

An attribute implication annotated by time points $A \Rightarrow B$ *is entailed by* a theory Σ if it is true in every model of Σ which we denote by $\Sigma \models A \Rightarrow B$.

Definition

Let $\mathcal{S}$ be a system of subsets of $\mathcal{T}_Y$. Then $\mathcal{S}$ is *closed under time shifts* if for every $i \in \mathbb{Z}$ and $M \in \mathcal{S}$ is also $M + i \in \mathcal{S}$.

Theorem

A set of all models of a theory is precisely an algebraic closure system closed under time shifts.

Canonical and finite representation

Definition

For $A, B \subseteq \mathcal{T}_Y$ we put $A \sqsubseteq B$ whenever there is $i \in \mathbb{Z}$ such that $A + i \subseteq B$; we put $A \not\sqsubseteq B$ if it is not the case that $A \sqsubseteq B$; we put $A \sqsubset B$ whenever $A \sqsubseteq B$ and $B \not\sqsubseteq A$; put $A \not\sqsubset B$ whenever $A \not\sqsubseteq B$ or $B \sqsubseteq A$; and put $A \equiv B$, whenever $A \sqsubseteq B$ and $B \sqsubseteq A$.

Definition

$$l(M) = \min\{i \in \mathbb{Z} \,|\, y^i \in M \text{ for some } y \in Y\},$$
$$u(M) = \max\{i \in \mathbb{Z} \,|\, y^i \in M \text{ for some } y \in Y\},$$
$$\|M\| = u(M) - l(M).$$

Theorem

If $\mathcal{S} \subseteq \mathcal{F}^*$ is finitely representable then $\mathcal{S}/_\equiv$ is a finite set. □

Input data

Definition

$$\mathcal{I} = \{I_x \in \mathcal{F} \mid x \in X\}.$$

Definition

$$A^{\uparrow_{\mathcal{I}}} = \bigcap\{I_x - i \mid x^i \in A\},$$
$$B^{\downarrow_{\mathcal{I}}} = \{x^i \mid B \subseteq I_x - i\}.$$

Definition

$$B^{\downarrow\uparrow} = \bigcap\{I_x - i \mid B \subseteq I_x - i\}.$$

Theorem

$\mathcal{M}_{\mathcal{I}} = \{M^{\downarrow\uparrow} \mid M \subseteq \mathcal{T}_Y\}$ is an algebraic closure system which is closed under time shifts. Moreover, we have $\mathcal{M}_{\mathcal{I}} \subseteq \mathcal{F}^*$ and $\mathcal{M}_{\mathcal{I}}$ is finitely representable.

Finitely generated bases

Definition

A theory Σ is *finitely generated* whenever $\mathrm{Mod}(\Sigma)$ is finitely representable, $\emptyset \in \mathrm{Mod}(\Sigma)$, and $\mathrm{Mod}(\Sigma) \cap \mathcal{F} \neq \emptyset$.

Theorem

Every Σ which is complete in $\mathcal{I}$ is finitely generated and for every finitely generated Γ there is input data $\mathcal{I}_\Gamma$.

Theorem

Let Σ be finitely generated theory. Then $\mathcal{S} = \{A \mid A \Rightarrow B \in \Sigma\}$ is not finitely representable.

Minimal bases

Definition

Σ is called complete in $\mathcal{I}$ whenever for every $A \Rightarrow B$ we have $\mathcal{I} \models A \Rightarrow B$ iff $\Sigma \models A \Rightarrow B$.

Definition

A set $P \in \mathcal{F}$ is a pseudo-intent of $\mathcal{I}$ if $P \neq P^{\downarrow\uparrow}$ and for any pseudo-intent Q of $\mathcal{I}$ such that $Q \subset P$ we have $Q^{\downarrow\uparrow} \subseteq P$. The set of all pseudo-intents of $\mathcal{I}$ is denoted by $\mathcal{P}_{\mathcal{I}}$.

Theorem

$\{P \rightarrow P^{\downarrow\uparrow} \mid P \in r(\mathcal{P})\}$ *is minimal and complete in* $\mathcal{I}$.