Fisher information and resolution beyond the Rayleigh limit

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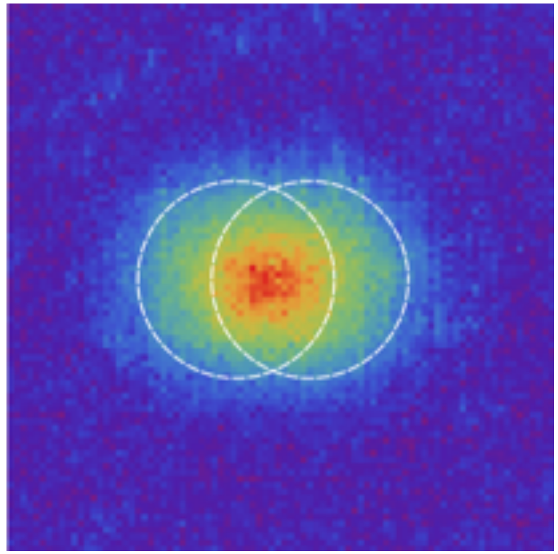
Outline

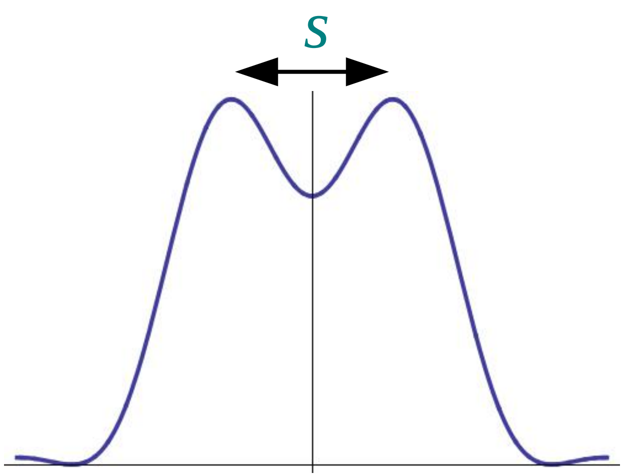
Background: Precision and Fisher information in optics

Quantum Fisher Information in general

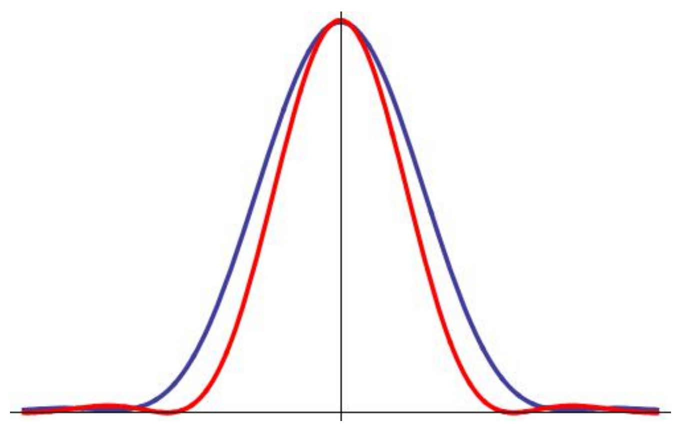
"Rayleigh limit" and two-point resolution

Optical resolution- Rayleigh criterion





standard resolution



super-resolution

Measurement and parameter estimation

Measurement: Born rule for (normalized) measurement on j-channel of transformed state

$$p_j(s) = \langle j|\rho(s)|j\rangle \qquad \rho(s) = U(s)^\dagger \rho U$$

$$A = \sum_j a_j |j\rangle\langle j| \qquad \Delta A = \left|\frac{\partial\langle A\rangle}{\partial s}\right|\Delta s$$

- Estimation: read-out of the parameter s from the registered values
- Variance of any unbiased estimation is limited by the Fisher Information (FI)
- Quantum Fisher Information (QFI) =Fisher information optimized over all possible detections

Fisher Information

$$\mathcal{F}_s = \mathbb{E}\left[\left(\frac{\partial \log p_n(s)}{\partial s}\right)^2\right] = \sum_n \frac{[p'_n(s)]^2}{p_n(s)}$$

Fisher information: limit for unbiased parameter estimation

$$\Delta s \geq 1\sqrt{nF}$$

Rayleigh curse

$$\mathcal{F}_s = \mathbb{E}\left[\left(\frac{\partial \log p_n(s)}{\partial s}\right)^2\right] = \sum_n \frac{[p_n'(s)]^2}{p_n(s)}$$

Fisher information for two point resolution: limit for unbiased parameter estimation

$$\Delta s \geq 1\sqrt{n\mathcal{F}}$$

$$p(x) = \frac{1}{2}[|\Psi(x+s)|^2 + |\Psi(x-s)|^2]$$
$$= I(x) + 1/2s^2 I''(x) + \ldots$$

$$\mathcal{F}_0 = s^2 \int dx \frac{I''(x)^2}{I(x)}$$

Quantum Fisher Information

For QFI, see the arguments of Helstrom 1975 ...

Optimize over all the measurement!!!

The necessary ingredient are symmetric logarithmic derivation expressed in diagonalizing basis.

$$\frac{\partial \rho}{\partial s} = 1/2(\mathcal{L}\rho + \rho\mathcal{L}) \qquad \rho = \sum \lambda_i |\varphi_i\rangle\langle\varphi_i|$$

$$\mathcal{F}_Q = Tr(\rho \mathcal{L}^2) = 2 \sum_{m,n} \frac{|\langle \varphi_n | \frac{\partial \rho}{\partial s} | \varphi_m \rangle|^2}{\lambda_n + \lambda_m}$$

Example: QFI for pure state

$$\rho(s) = |\Psi(s)\rangle\langle\Psi(s)|$$

$$\mathcal{F}_Q = 4\langle\Psi(s)|(\frac{\partial \rho}{\partial s})^2|\Psi(s)\rangle$$

Zero eigenvalues cannot be neglected but eliminated !

Problems of QFI: large ambiguity as far measurement is concerned, optimality many aspects…

Two-point resolution

$$\varrho_s = q\,|\Psi_+\rangle\langle\Psi_+| + (1-q)\,|\Psi_-\rangle\langle\Psi_-|$$

$$|\Psi_\pm\rangle = e^{\pm isP/2}|\Psi\rangle$$

- FI and QFI for two-point resolution: Tsang 2016
- Here: optical arguments and symmetry arguments" for optimal measurement achieving QFI

Symmetry for achieving QFI

Assume symmetry of the point-spread- function as well as the symmetry of the measurement

$$\Psi(x) = \Psi(-x) \qquad \langle x|n\rangle = \pm\langle -x|n\rangle$$

The measurement does not feel the two-component structure of the signal! The original two-point resolution problem has been effectively transformed to localization of a single point source.

$$p_n \equiv |a_n|^2 = |\langle n|\Psi_\pm\rangle|^2$$

QFI can be obtained from FI just by expressing probabilities by complex amplitudes ...

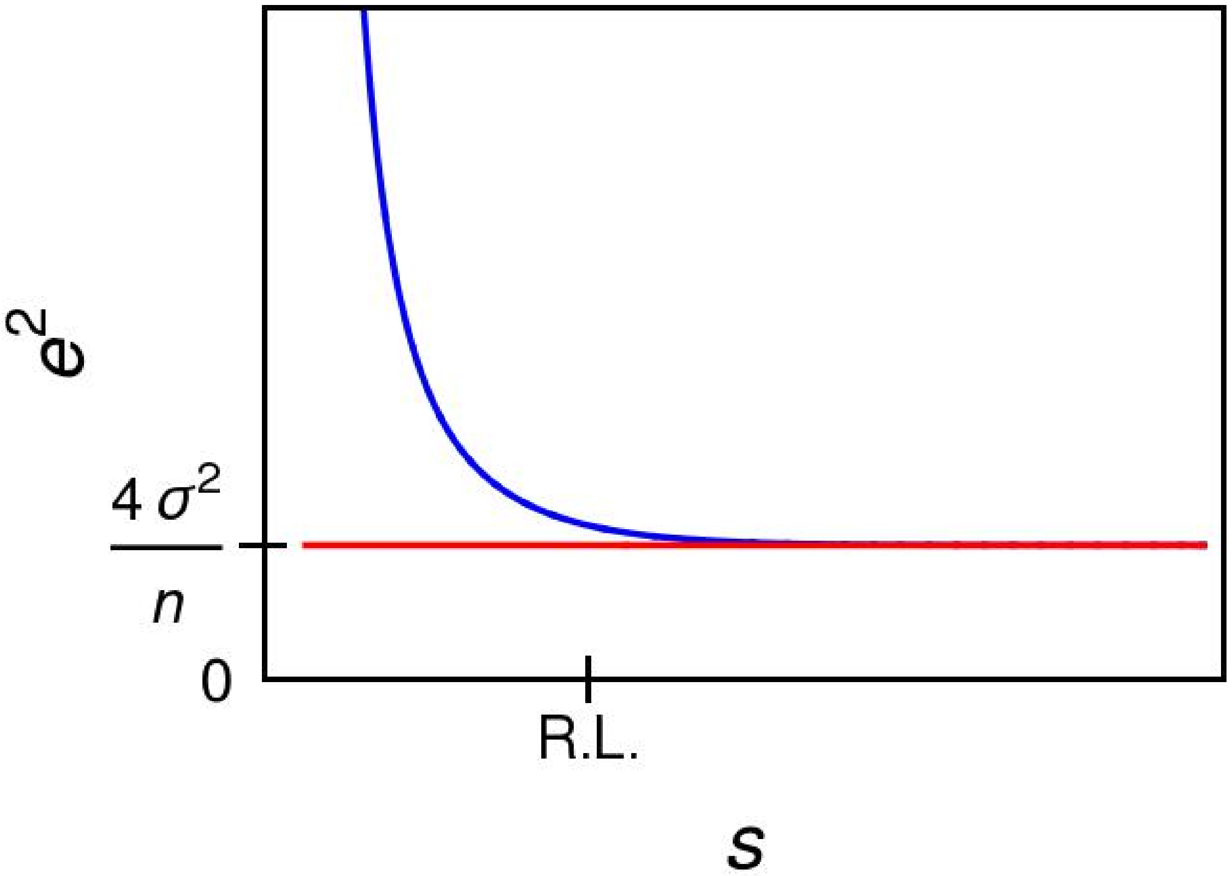
$$\mathcal{F} = \sum_n \frac{[p_n'(s)]^2}{p_n(s)}$$

$$= 4 \sum_n |\frac{\partial a_n}{\partial s}|^2 + \sum_n \frac{1}{p_n}[a_n^* \frac{\partial a_n}{\partial s} - a_n \frac{\partial a_n^*}{\partial s}]^2$$

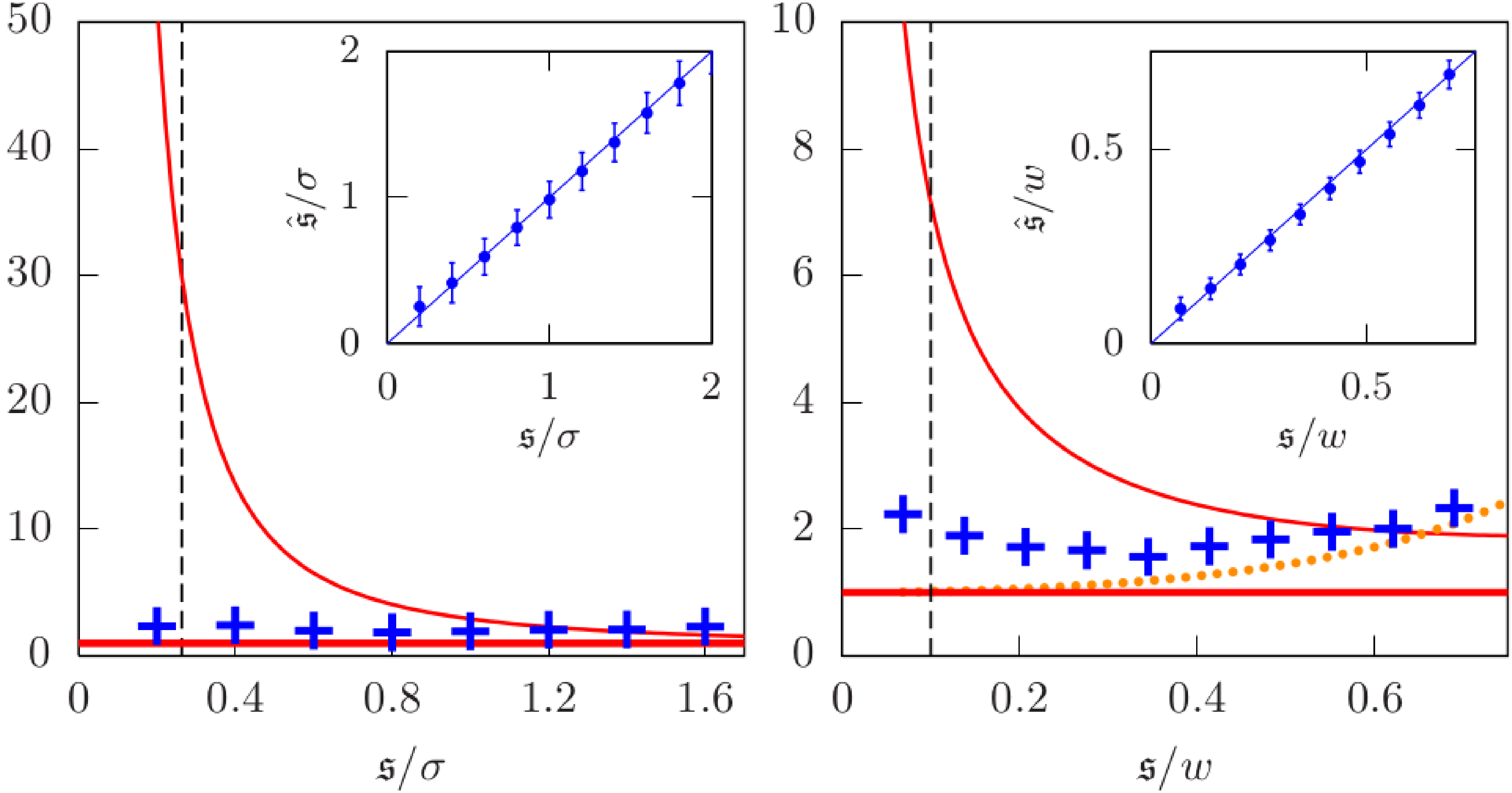
Optimality conditions:

$$\mathrm{Im}\left(a_n \frac{\partial a_n^*}{\partial s}\right) = 0$$

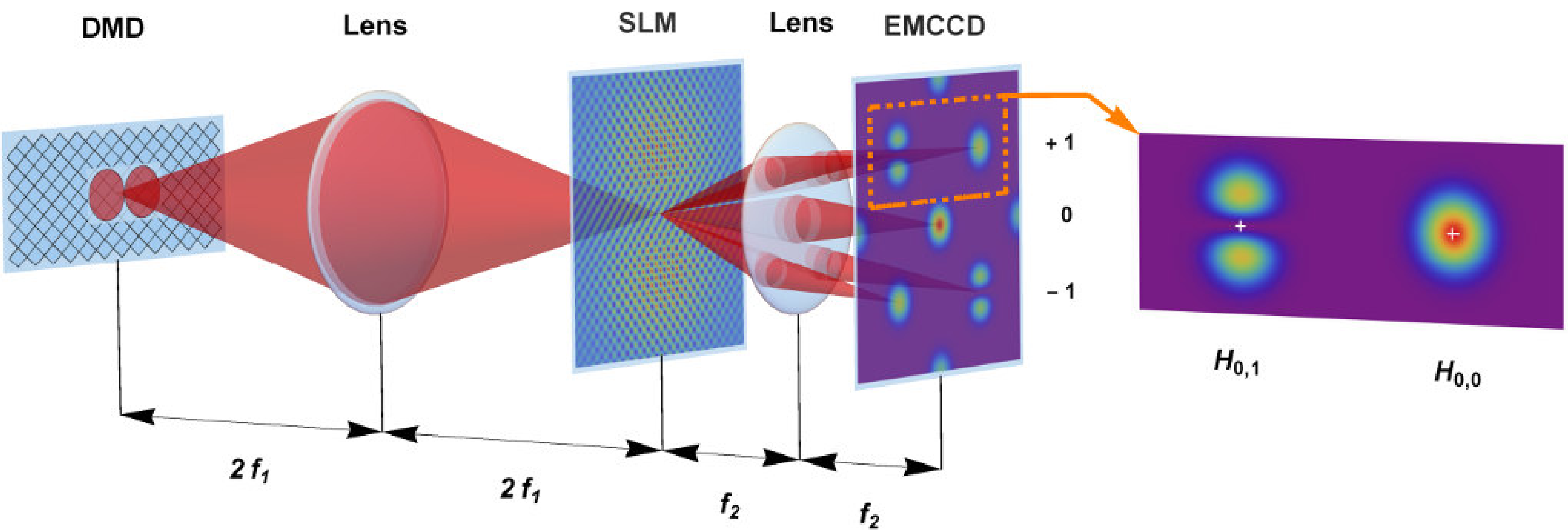
FI vs FQI



Results: Gaussian and sinc PDF



Experimental setup



Measurement achieving FQI

There is an ambiguity how to fulfill the optimality conditions. The ultimate resolution should not be considered as a rarity, but rather as a feature shared by many permissible detection schemes.

Efficiency vs. universality

How to do the detection efficiently?

Suggestion: Project the signal on a set of orthonormalized derivatives of $\Psi(x)$-PSF adapted schemes

$$\Phi_n(p) \equiv \langle p|n\rangle = Q_n(p)\Psi(p)$$

$$\Phi_n(x) \equiv \langle x|n\rangle = \frac{1}{\sqrt{2\pi}}\int Q_n(p)\Psi(p)e^{ipx}$$

Example 1: Gaussian PSF

$$\Psi(x) = (2\pi)^{-1/4} \exp(-x^2/4), \quad \sigma = 1$$

The optimal PSF-adapted set :

Hermite-Gauss modes

$$\mathcal{F}_s = 1/4$$

Example 2: Sinc PSF

$$\Psi(x) = \frac{1}{\sqrt{\pi}}\mathrm{sinc}(x), \;\; \Psi(p) = \frac{1}{\sqrt{2}}\mathrm{rect}(p/2)$$

The optimal PSF-adapted set is linked with

Legendre polynomials orthogonal on (-1/2,1/2)

$$a_n = \langle n | \Psi_{\pm} \rangle = \frac{\sqrt{2n+1}}{2} \int_{-1}^{1} L_n(p)\, e^{-isp/2}\, dp$$

Example 2: Sinc PSF…

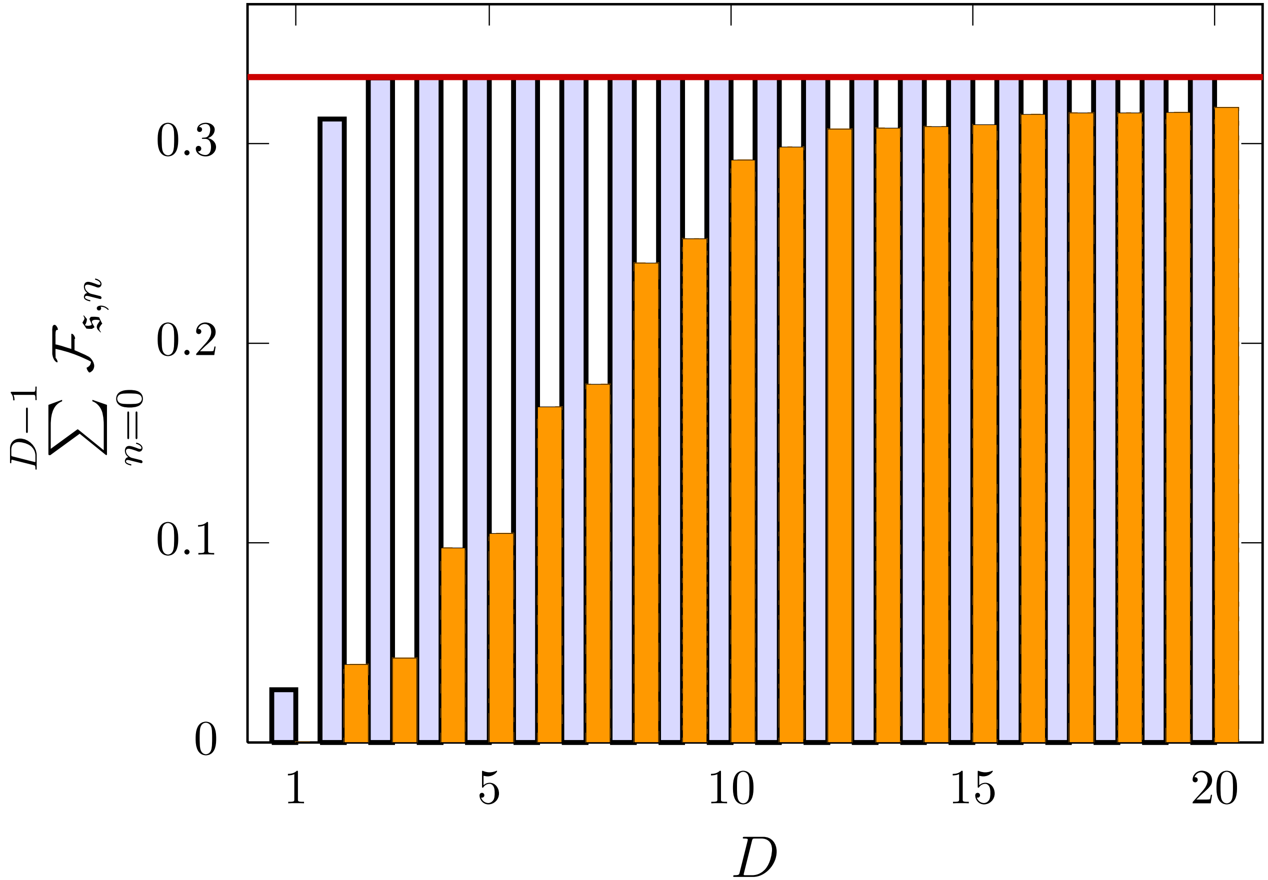
Efficient measurement modes:

$$\Phi_n(x) = \sqrt{n+1/2}\,\frac{J_{n+\frac{1}{2}}(x)}{\sqrt{x}}$$

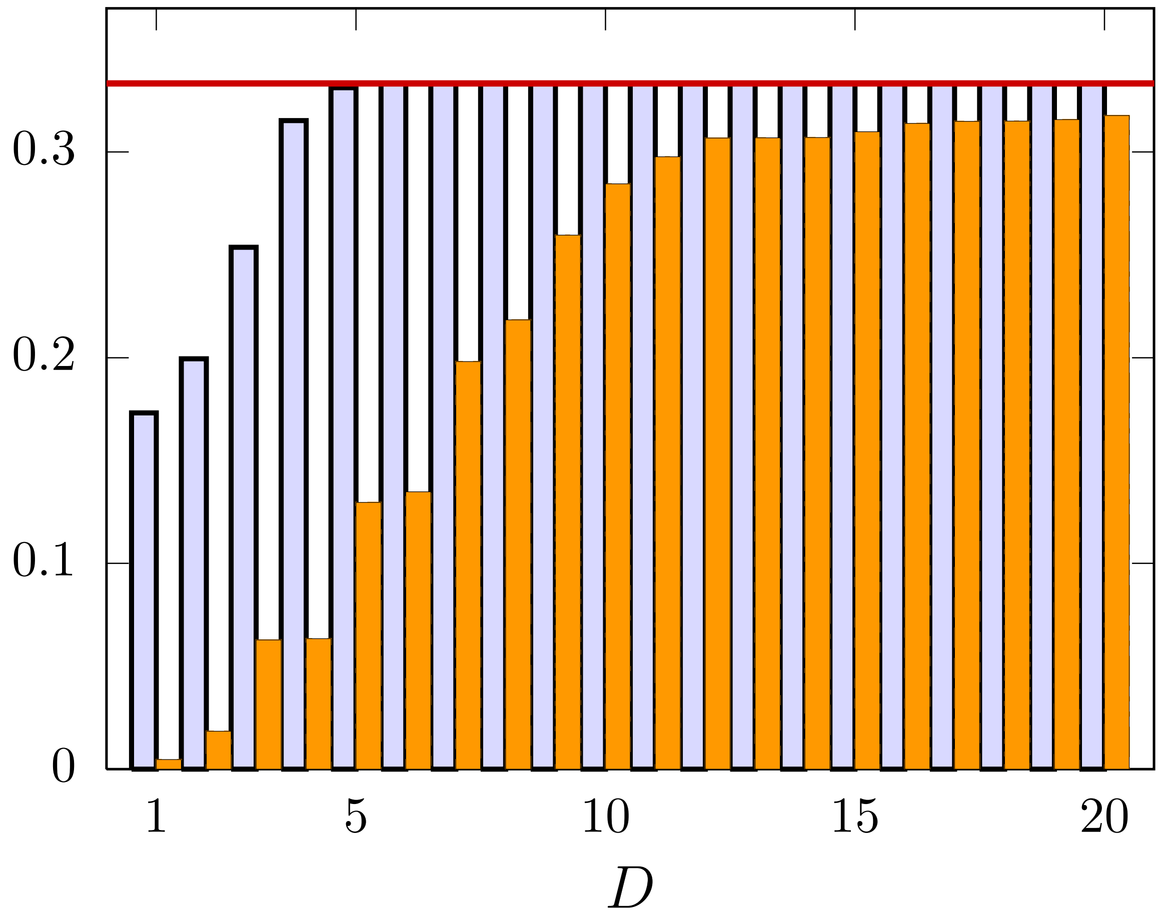
Fisher information consists of partial contributions:

$$\mathcal{F}_{s,n} = \frac{\pi\left[nJ_{n-\frac{1}{2}}(s/2) - (n+1)J_{n+\frac{3}{2}}(s/2)\right]^2}{(2n+1)s}$$

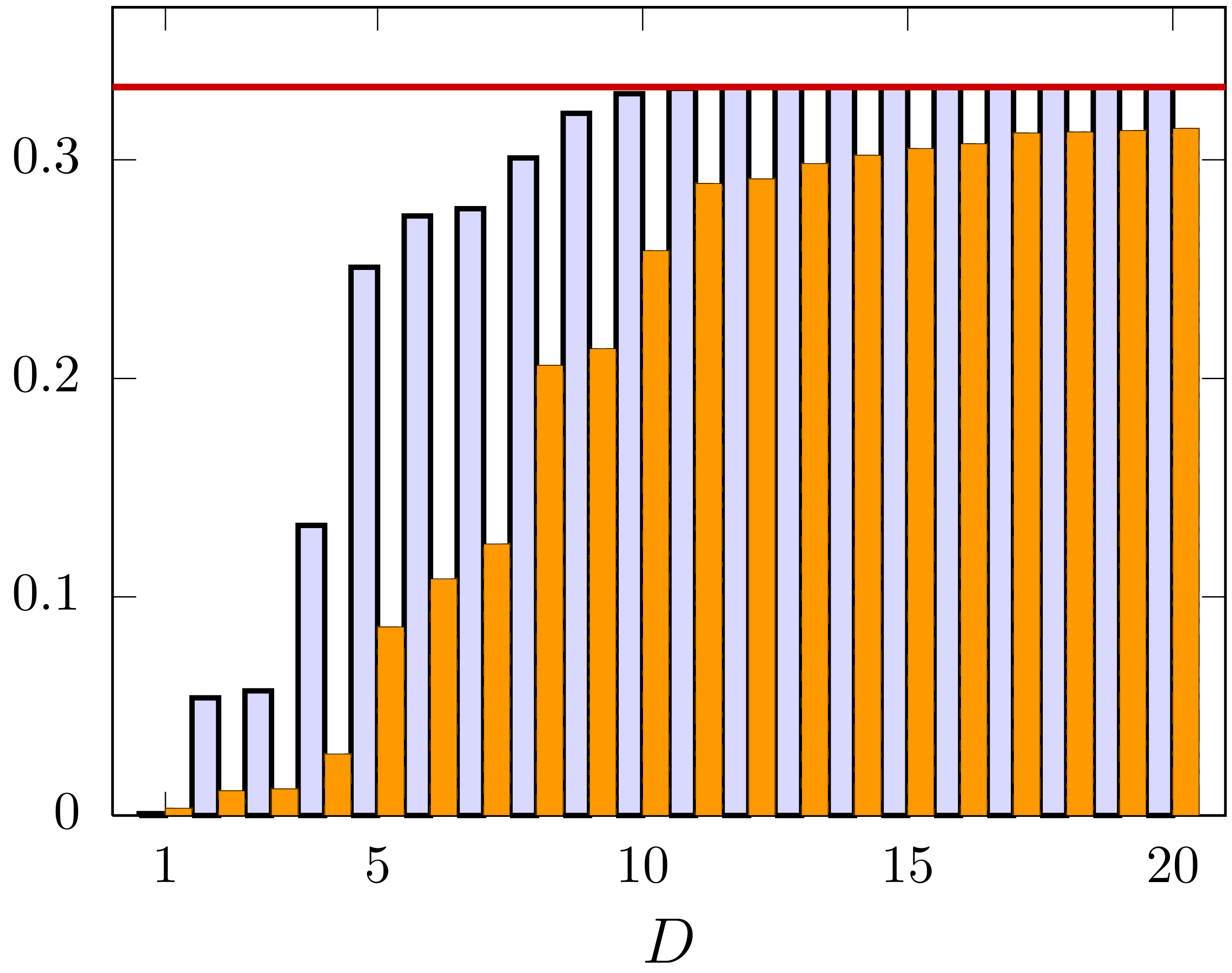
$$\mathcal{F}_s = 1/3$$



FI for the first D projections on the HG basis with arbitrarily chosen σ = π (orange bars) and the PSF Sinc adapted measurement, **Separation s= 1, Rayleigh limit = π**. More than a hundred of Hermite-Gauss projections must be measured to access 98.5% of the QFI (horizontal red line), whereas just three projections of the PSF-adapted measurement are sufficient.



As before, **Separation s= 2, Rayleigh limit = π**



As before, **Separation s= 15,Rayleigh limit = π**

Towards Realistic Superresolution

J. Rehacek, Z. Hradil, B. Stoklasa, M. Paur, J. Grover, A. Krzic, L. L. Sanchez-Soto Multiparameter Quantum Metrology of Incoherent Point Sources: Towards Realistic Superresolution, arXiv:1709.07705

$$\rho_\theta = \mathfrak{q}\,\rho_+ + (1-\mathfrak{q})\,\rho_- \,,$$

$$|\Psi_\pm\rangle = \exp[-i(\mathfrak{s}_0 \pm \mathfrak{s}/2)P]|\Psi\rangle \,,$$

QFI matrix

$$Q_{\alpha\beta}(\boldsymbol{\theta}) = 2\sum_{m,n} \frac{1}{\lambda_m + \lambda_n} \langle \lambda_m | \partial_\alpha \rho_{\boldsymbol{\theta}} | \lambda_n \rangle \langle \lambda_n | \partial_\beta \rho_{\boldsymbol{\theta}} | \lambda_m \rangle$$

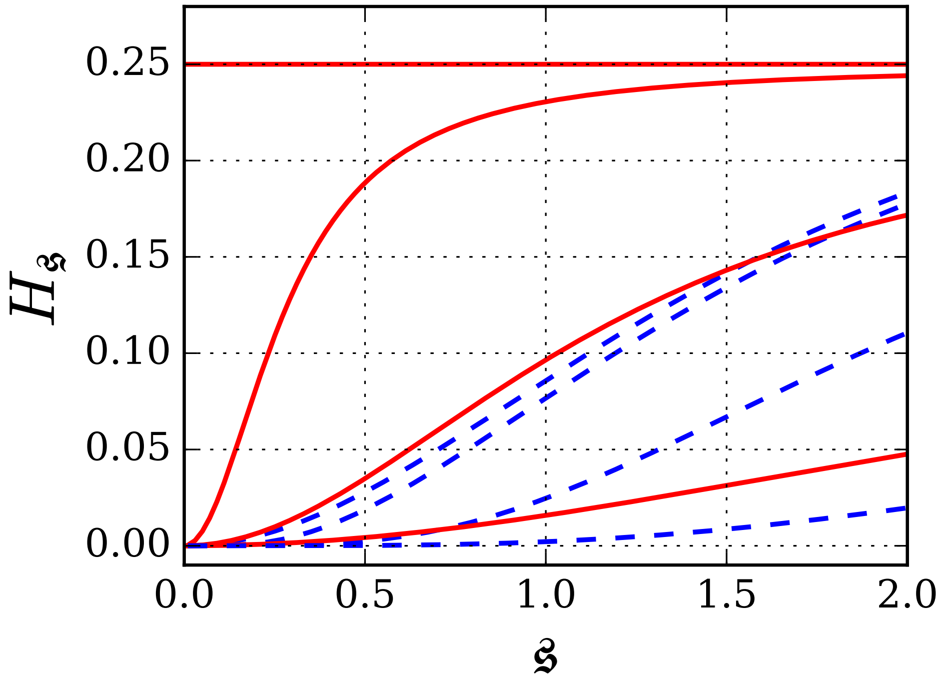
$$Q = 4 \begin{pmatrix} p^2 + 4\mathfrak{q}(1-\mathfrak{q})\wp^2 & (\mathfrak{q} - 1/2)p^2 & -iw\wp \\ (\mathfrak{q} - 1/2)p^2 & p^2/4 & 0 \\ -iw\wp & 0 & \dfrac{1-w^2}{4\mathfrak{q}(1-\mathfrak{q})} \end{pmatrix}$$

$$w \equiv \langle \Psi_\pm | \Psi_\mp \rangle = \langle \Psi | \exp(i\mathfrak{s}P) | \Psi \rangle ,$$

$$p^2 \equiv \langle \Psi_\pm | P^2 | \Psi_\pm \rangle = \langle \Psi | P^2 | \Psi \rangle ,$$

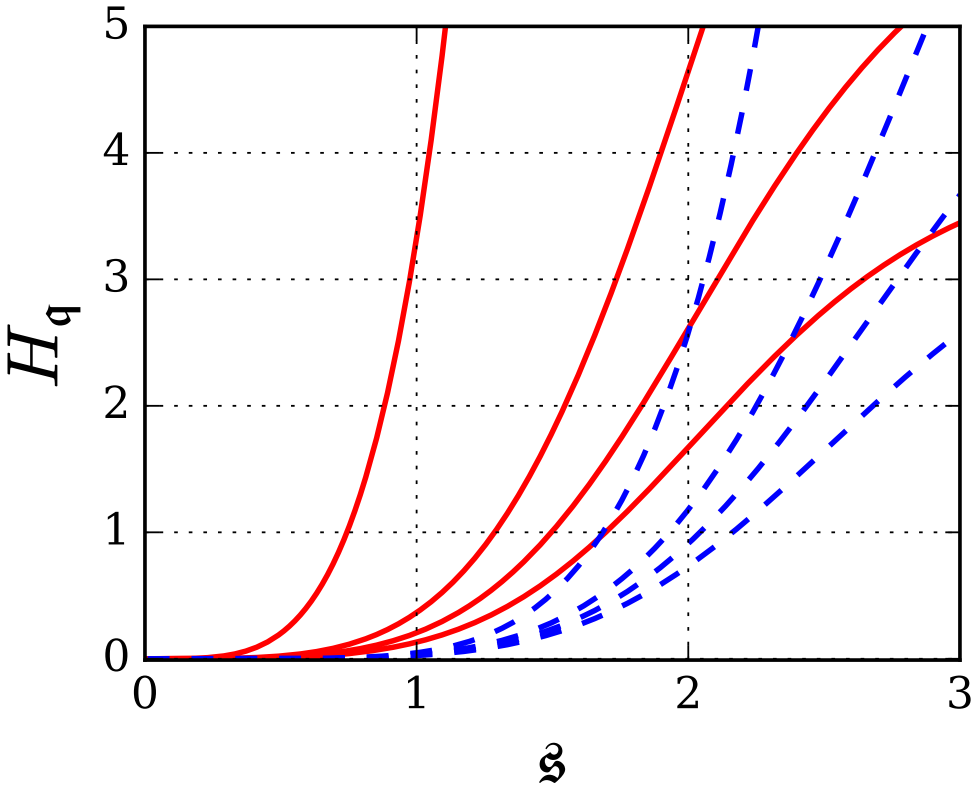
$$\wp \equiv \pm \langle \Psi_\pm | P | \Psi_\mp \rangle = \langle \Psi | \exp(i\mathfrak{s}P) P | \Psi \rangle$$

Precision for separation



The values of q, from top to bot- tom, are 0.5, 0.45, 0.3, and 0.1. Notice that the performance of the optimal detection is rather sensitive to small deviations from equal brightness over a wide range of separations.

Precision for intensities



Precision about relative intensity q as inferred by the optimal detection (red solid lines) and the direct detection (blue broken lines) for different relative intensities of the two sources. The values of q, from bottom to top are 0.5, 0.2, 0.1, 0.01.

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Wavefront sensing reveals optical coherence

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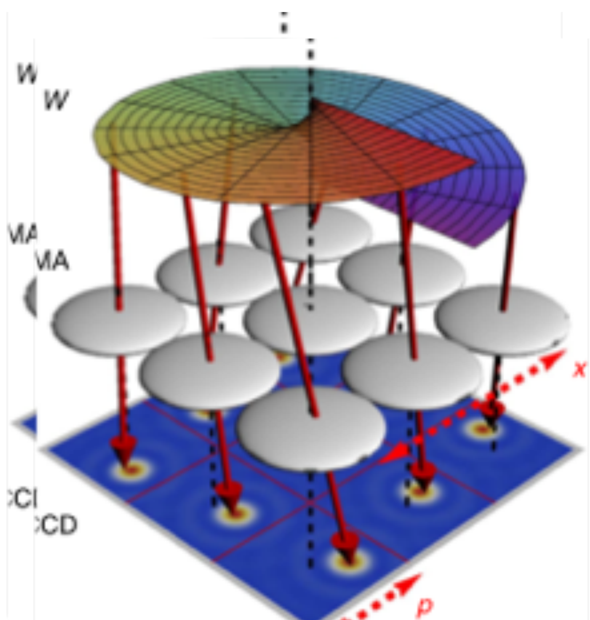

Fisher Info Matrix provides a useful tool for assessing the performance of reconstruction schemes

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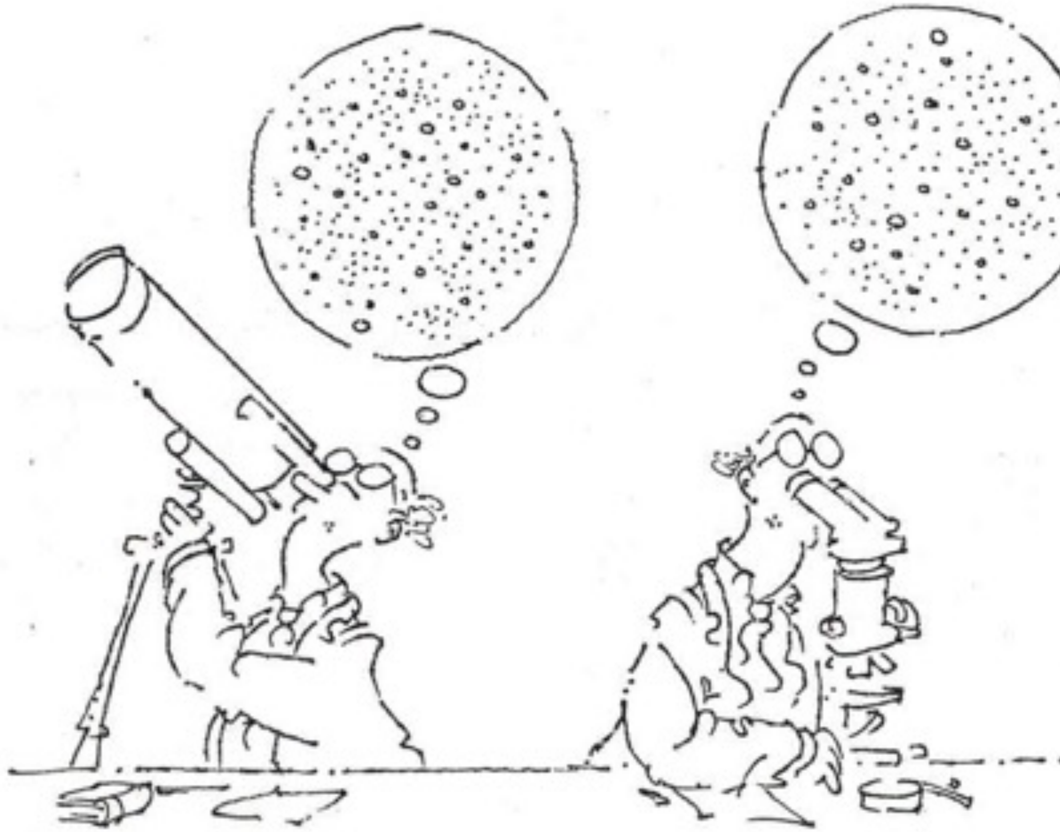

Thanks for your attention!